The Hochschild–Serre Spectral Sequence

Concise notes ,
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Contents

Chapter 1

Spectral Sequences

1.1 Spectral Sequences

Definition 1.1.1. A **cohomological spectral sequence** consists the following data:

i. A family $(E_r^{p,q})$ for $p, q \in \mathbf{Z}$ and $r \geq 0$. For a fixed r the collection $(E_r^{p,q})_{p,q\in\mathbb{Z}}$ is called the r**-th** page of the spectral sequence.

ii. Differentials: $\mathrm{d}_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ such that $\mathrm{d}_r \circ \mathrm{d}_r = 0$, and more precisely:

$$E^{p-r,q+r-1} \xrightarrow{\mathrm{d}_r} E_r^{p,q} \xrightarrow{\mathrm{d}_r} E_r^{p+r,q-r+1} = 0$$

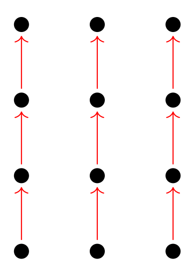
iii. Isomorphisms: For all $r \geq 0$ we have that $E_{r+1}^{p,q} \cong H^{p,q}(E_r)$ meaning:

$$E_{r+1}^{p,q} = \frac{\ker\left(E_r^{p,q} \xrightarrow{\mathrm{d}_r} E_r^{p+r,q-r+1}\right)}{\operatorname{Im}\left(E_r^{p-r,q+r-1} \xrightarrow{\mathrm{d}_r} E_r^{p,q}\right)}.$$

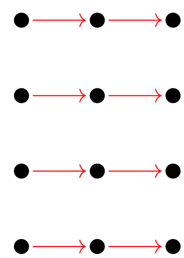
I.e. the $r+1$-th page can be obtained by taking the cohomology of the r-th page.

Let's try to visualise the first pages of a spectral sequence:

$$\text{0-page:}\mathrm{d}_0 : E_0^{p,q} \longrightarrow E_0^{p,q+1}$$



$$\text{1st page } \mathrm{d}_1 : E_1^{p,q} \longrightarrow E_1^{p+1,q}$$



The interesting things start when we start to study the 2nd page and after where $\mathrm{d}_2 : E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$

Similarly we get the homological definition of a spectral sequence, which is denoted by $\left(E_{p,q}^r\right)$, for $p, q \in \mathbf{Z}$ and $r \geq 0$ and we have analogous need of considering differentials

$$\mathrm{d}^r : E_{p,q}^r \longrightarrow E_{p-r,q+r-1}^r, \text{ where } \mathrm{d}^r \circ \mathrm{d}^r = 0$$

and the $r+1$ - is obtained by taking the homology of the r - the page in an analogous way as the cohomological definition.

1.2 Graded and Bigraded Modules and their Filtrations

Definition 1.2.1. A **graded module** over a ring R is a family $M_* = (M_n)_{n\in\mathbb{Z}}$.

Example 1.2.1. Given a complex C_* of R-modules, the family $C_* = (C_n)_{n\in\mathbb{Z}}$ is a graded module. Similarly, considering the homology of this complex gives us $H(C_*) = (H_n(C_*))_n$ which is a graded module.

Definition 1.2.2. Let M and N two graded modules. A **graded map** of degree d is a family

$$f = \left(M_n \xrightarrow{f_n} N_{n+d}\right)_{n\in\mathbb{Z}}.$$

Example 1.2.2. i. Let C_* be any chain complex then each differential d_n is $\mathrm{d}_n : C_n \longrightarrow C_{n-1}$ so the family $\mathrm{d} = (\mathrm{d}_n : C_n \longrightarrow C_{n-1})$ is a graded map of degree -1.

ii. If C_*, D_* are two complexes, a chain morphism $f : C_* \to D_*$ is a graded map of degree 0.

iii. A null homotopic map for a given $f : C_* \to D_*$ is a graded map of degree 1.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & \swarrow{\scriptstyle \varphi_n} & \downarrow{\scriptstyle f_n} & \swarrow{\scriptstyle \varphi_{n-1}} & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots
\end{array}$$

Look that $\varphi = (C_n \longrightarrow D_{n+1})_{n\in\mathbb{Z}}$ such that $f_n = \mathrm{d}_{n+1}^D \varphi_n + \varphi_{n+1} \mathrm{d}_n^C$

Definition 1.2.3. The category of **graded modules** over R has objects all graded modules and morphisms all graded maps of graded modules over R.

Definition 1.2.4. Let M, N be two graded modules over R. We say that that N is a submodule of M if $N_n \subseteq M_n$ for all $n \in \mathbb{Z}$. The quotient of M by N is defined as

$$\frac{M}{N} = \left(\frac{M_n}{N_n}\right)_{n\in\mathbb{Z}}.$$

For a graded map, we have a natural way to define the concepts of kernel and image as follows : For a graded map $f = (f_n : M_n \longrightarrow N_{n+d})_{n\in\mathbb{Z}}$. The kernel of f and the image of f are the graded modules

$$\ker f := (\ker f_n)_{n\in\mathbb{Z}} \quad \text{and} \quad \operatorname{Im} f := (\operatorname{Im} f_{n-d})_n .$$

We say that a sequence of graded modules $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if $\operatorname{Im} f = \ker g$.

Definition 1.2.5. Let R be a ring. A **bigraded module** $M_{\bullet,\bullet}$ is $M_{\bullet,\bullet} = (M_{p,q})_{p,q\in\mathbb{Z}}$. Similarly, a **bigraded morphism** of degree (α, β) between two bigraded modules M, N is;

$$f = \left(M_{p,q} \xrightarrow{f_{p,q}} N_{p+\alpha,q+\beta}\right)_{p,q\in\mathbb{Z}}.$$

Similarly we define

$$\ker(f) = (\ker f_{p,q})_{p,q\in\mathbb{Z}} \quad \text{and} \quad \operatorname{Im}(f) = (\operatorname{Im} f_{p-\alpha,q-\beta})_{p,q\in\mathbb{Z}} .$$

Example 1.2.3. Let $E^r_{p,q}$ be a homological spectral sequence for fixed $r \geq 0$. We have $E^r = \left(E^r_{p,q}\right)$ is a bigraded module and the differentials $\mathrm{d}^r = \left(E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}\right)_{p,q}$ are bigraded morphisms of degree $(-r, r-1)$.

Example 1.2.4. Similarly, we can define the (co)homology of degree (p, q) for a given differential

$$M = (M_{p,q})_{p,q}, \qquad d = (f_{p,q} : M_{p,q} \to M_{p+\alpha,q+\beta})$$

$$H_{p,q}(M_\bullet) = \frac{\ker\left(f_{p,q} : M_{p,q} \to M_{p+\alpha,q+\beta}\right)}{\operatorname{Im}\left(f_{p-\alpha,q-\beta} : M_{p-\alpha,q-\beta} \to M_{p,q}\right)}.$$

The homology of (M, d) is denoted by $H(M, \mathrm{d})$. Note, in the case of spectral sequences $H(E^r, \mathrm{d}^r) = E^{r+1}$.

Warning !

The homology of a given differential is not always differential. However, in the case of spectral sequences is (!) by definition.

1.3 Filtrations

Definition 1.3.1. $M \in R$-Mod. A **filtration** of M is a family of submodules: $F^\bullet M = (F^p M)$ such that $F^p M \subseteq F^{p+1} M$. In this case we say that the filtration is **increasing**. Similarly, if $F^{p+1} M \subseteq F^p M$ we say that the filtration is decreasing.

If the filtration is increasing, the **factors** of $F^\bullet$ are $F^{p+1}M/F^pM$.

Remark 1.3.1. $F^\bullet M = (F^p M)_{p \in \mathbf{Z}}$ is a graded module on $\mathbb{Z}$ and in the increasing case, the induced family of inclusions $(F^p M \to F^{p+1} M)_{p \in \mathbb{Z}}$ is an 1-graded map

Now we take a graded module $M_\bullet = (M_n)_{n \in \mathbb{Z}}$. We expect a filtration of a graded module to be a bigraded module which provides a filtration to each individual term such as : $F^\bullet(M_\bullet) = (F^p M_n)_{p,n}$ such that

$$F^p(M_n) \subseteq F^{p+1}(M_n), \quad \text{for all } n \in \mathbf{Z} \text{ and for all } p \in \mathbf{Z}$$

i.e. $F^p(M_\bullet) \subseteq F^{p+1}(M_\bullet)$. **What changes if (M, d) is a differential ?** We now consider differential, so that the objects of this family to be connected by. A reasonable definition for a filtration of a differential is to connect the objects of the the induced graded module $F^p(M_\bullet)$, for all $p \in \mathbf{Z}$. For example we assume that d is of degree 1. We would like :

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M_{n+1} & \xrightarrow{\mathrm{d}} & M_n & \xrightarrow{\mathrm{d}} & M_{n-1} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & F^{p+1}M_{n+1} & \xrightarrow{\mathrm{d}^{p+1}_{n+1}} & F^{p+1}M_n & \xrightarrow{\mathrm{d}^{p+1}_{n}} & F^{p+1}M_{n-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & F^{p}M_{n+1} & \xrightarrow{\mathrm{d}^{p}_{n+1}} & F^{p}M_n & \xrightarrow{\mathrm{d}^{p}_{n}} & F^{p}M_{n-1} & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where d^p_n is the restriction of the starting differential to the submodule $F^p M_n$. So in this case, we add one more condition:

$$\mathrm{d}_n \left(F^p(M_n)\right) \subseteq F^p(M_{n-1})$$

i.e. for each $p \in \mathbb{Z}$, the family $(F^p M_\bullet)$ is a differential, restricting the starting one. So we get a filtration of sub-complexes:

$$\cdots \subseteq F^{p-1}C_\bullet \subseteq F^p C_\bullet \subseteq F^{p+1} C_\bullet \subseteq \cdots$$

We called this a **filtration of the complex** $(C_\bullet, \mathrm{d})$, and we denote by $(C_\bullet, d, F)$.

Remark 1.3.2. Let $(C_\bullet, \mathrm{d}, F)$ be a filtered complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

We saw that for each $p \in \mathbf{Z}$ a new chain complex is induced :

$$\cdots \longrightarrow F^p(C_{n+1}) \xrightarrow{\mathrm{d}_{n+1}} F^p(C_n) \xrightarrow{\mathrm{d}_n} F^p(C_{n-1}) \xrightarrow{\mathrm{d}_{n-1}} \cdots$$

We notice that $\ker \mathrm{d}_n^p \subseteq \ker \mathrm{d}_n$ and $\mathrm{Im}\, \mathrm{d}_{n+1}^p \subseteq \mathrm{Im}\, \mathrm{d}_n$. Then, there is a map :

$$H_n(F^p C_\bullet) = \frac{\ker \mathrm{d}_n^p}{\mathrm{Im}\, \mathrm{d}_{n+1}^p} \to H_n(C_\bullet) = \frac{\ker \mathrm{d}_n}{\mathrm{Im}\, \mathrm{d}_{n+1}},$$

So we have an induced filtration on the homology:

$$F^p H_n(C_\bullet) = \mathrm{Im}\left(H_n(F^p C_\bullet) \to H_n(C_\bullet)\right)$$

which gives us a filtration on the graded module $H(C_\bullet)$. A reasonable question is whether or not

$$F^p H_n(C_\bullet) = H_n\left(F^p C_\bullet\right)$$

For the following important applications that we are going to present in the next chapter we are mostly interested on filtrations which sometime stop, i.e. there is an upper and a lower bound of them.

Definition 1.3.2. We say that a filtration of a graded module M is **bounded** if for all $n \in \mathbb{N}$, there are $s(n), t(n) \in \mathbb{Z}$ with $s(n) \leq t(n)$ such that :

$$F^s(M_n) = 0 \quad \text{and} \quad F^t(M_n) = M_n$$

i.e.

$$F^i(M_n) = 0, \quad \text{for all } i \leq s \quad \text{and} \quad F^j(M_n) = M_n, \quad \text{for all } j \geq t.$$

Warning !

In this definition we don't assume that s, t are global! i.e that there are $s, t \in \mathbb{Z}$,

$$F^s(M_n) = 0 \quad \text{and} \quad F^t(M_n) = M_n, \quad \text{for all } n \in \mathbb{Z}.$$

How does a bounded complex filtration affects the induced filtration on homology? Obviously, if for all n, there are $s(n), t(n) \in \mathbb{Z}$:

$$F^s(M_n) = 0 \quad \text{and} \quad F^t(M_n) = M_n,$$

we recall that if $i_p : F^p(C_\bullet) \hookrightarrow C_\bullet$ then the induced filtration on homology is defined by

$$F^p(H_n(C_\bullet)) = \mathrm{Im}\left(H_n(F^p C_\bullet) \to H_n(C_\bullet)\right).$$

If $p \leq s$, then $F^p(M_n) = 0$ so $H_n(F^p(C_\bullet)) = 0$, hence $F^p(H_n(C_\bullet)) = 0$. If $t \leq p$, then $F^p M_n = M_n$, so $F^p H_n(C_\bullet) = H_n(C_\bullet)$. So, the induced filtration on the homology is bounded.

Definition 1.3.3. Let M be a module or a graded module, and F be an increasing filtration of M, i.e:

$$\cdots \subseteq F^p(M) \subseteq F^{p+1}(M) \subseteq \cdots$$

The **associated graded module** is

$$\mathrm{gr}_F(M) = \left(\frac{F^{p+1}(M)}{F^p(M)}\right)_{p\in\mathbb{Z}}.$$

Remark 1.3.3. If M is graded, we have that for all $n \in \mathbb{N}$: $\cdots \subseteq F^p(M_n) \subseteq F^{p+1}(M_n) \subseteq \cdots$ so each quotient is defined as follows :

$$\frac{F^{p+1}(M)}{F^p(M)} := \left(\frac{F^{p+1}(M_n)}{F^p(M_n)}\right)_{n\in\mathbf{Z}}$$

Exercise 1.3.1. Define the associated graded module in the case of decreasing filtrations.

A filtered graded module $M_\bullet$ induces a graded module, $\mathrm{gr}_F(M) = (F^{p+1}(M)/F^p(M))_{p\in\mathbb{Z}}$ where $F^{p+1}(M)$ is graded. If (M, d) is a differential graded module, we have a graded morphism

$$\mathrm{d} = \left(M_n \xrightarrow{\mathrm{d}_n} M_{n+\alpha}\right)_{n\in\mathbb{Z}}$$

Do we expect the same thing happens for gr_F ? In particular, for all $n \in \mathbb{Z}$, and for all $p \in \mathbb{Z}$:

$$\widetilde{\mathrm{d}}_n : \frac{F^{p+1}(M_n)}{F^p(M_n)} \longrightarrow \frac{F^{p+1}(M_{n+\alpha})}{F^p(M_{n+\alpha})} \ ?$$

Let's look carefully what happens in the case of complexes. Let $(C_\bullet, d)$ a complex, and F be a filtration on $(C_\bullet, d)$, i.e.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\mathrm{d}_{n+1}} & C_n & \xrightarrow{\mathrm{d}_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & F^pC_{n+1} & \xrightarrow{\mathrm{d}_{n+1}} & F^pC_n & \xrightarrow{\mathrm{d}_n} & F^pC_{n-1} & \xrightarrow{\mathrm{d}_{n-1}} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & F^{p-1}C_{n+1} & \xrightarrow{\mathrm{d}_{n+1}} & F^{p-1}C_n & \xrightarrow{\mathrm{d}_n} & F^{p-1}C_{n-1} & \xrightarrow{\mathrm{d}_{n-1}} & \cdots
\end{array}$$

Looking the following composition :

$$F^pC_{n+1} \xrightarrow{\mathrm{d}_{n+1}} F^pC_n \longrightarrow \frac{F^p(C_n)}{F^{p-1}(C_n)}, \quad x \mapsto \mathrm{d}_{n+1}(x) \mapsto \overline{\mathrm{d}_{n+1}(x)}$$

However, by definition we have $\mathrm{d}_{n+1}\left(F^p(C_{n+1})\right) \subseteq F^p(C_n)$ which gives a morphism :

$$\frac{F^p(C_{n+1})}{F^{p-1}(C_{n+1})} \xrightarrow{\widetilde{\mathrm{d}}_{n+1}} \frac{F^p(C_n)}{F^{p-1}(C_n)}$$

We can easily check that $\widetilde{\mathrm{d}}_n \circ \widetilde{\mathrm{d}}_{n+1} = 0$.

Example 1.3.1. We can filter $\mathbb{Z}$ as $\mathbb{Z}$-module as follows:

$$\begin{cases} F^n(\mathbb{Z}) = \mathbb{Z}, & \text{for all } n \leq 0, \\ F^n(\mathbb{Z}) = 2^n\mathbb{Z}, & \text{for all } n > 0. \end{cases}$$

What about the graded associated module? For $n \leq 0$:

$$F^n(\mathbb{Z}) = \mathbb{Z} \quad \Rightarrow \quad \mathrm{gr}_F(\mathbb{Z})_n = F^n(\mathbb{Z})/F^{n+1}(\mathbb{Z}) = 0.$$

If $n > 0$ we have

$$\mathrm{gr}_F(\mathbb{Z})_n = \frac{2^n\mathbb{Z}}{2^{n+1}\mathbb{Z}} \cong \mathbb{Z}_2.$$

Example 1.3.2. Given if the $\mathrm{gr}_F(M)$ gives us useful information about M, is it sufficient for fully recover the module M? Let module M and a filtration F of M as follows:

$$F^pM = \begin{cases} 0, & p < 0, \\ A, & p = 0, \\ B, & p > 0. \end{cases}$$

i.e.

$$0 \subseteq 0 \subseteq \cdots \subseteq 0 \subseteq A \subseteq B \subseteq B \subseteq \cdots$$

So

$$\mathrm{gr}_F(M) = \begin{cases} 0, & p \neq 0, 1, \\ A, & p = 0, \\ B/A, & p = 1. \end{cases}$$

Since we only know about A, $\frac{B}{A}$, using the short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow \frac{B}{A} \longrightarrow 0$$

we can't giving full information for B since may $\mathrm{Ext}^1\left(\frac{B}{A}, A\right) \neq 0$, so may this sequence does not split.

1.4 Convergence of Spectral Sequences

1.4.1 Bounded and First Quadrant Spectral Sequences

Definition 1.4.1. a. We say that a cohomological spectral sequence $(E_r^{p,q})$ is **bounded** if for every n there are finitely many terms of total degree n, meaning that there are only finitely many pairs (p, q) with $n = p + q$ such that $E_r^{p,q} \neq 0$, for all $r \geq 0$.

b. Also, $E_r^{p,q}$ is called **first quadrant** if $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$.

Exercise 1.4.1. Give the analogous definition for the cohomological case.

Let's analyze carefully the definition given above. Fix (p, q). Then the differential at the r - th page are :

$$\underbrace{E^r_{p+r,q-r+1}}_{(1)} \xrightarrow{d^r} E^r_{p,q} \xrightarrow{d^r} \underbrace{E^r_{p-r,q+r-1}}_{(2)}$$

by changing pages may the indices of (1) and (2) change. However, the total degree of them is constant and

$$n_1 = (p + r) + (q - r + 1) = p + q + 1$$

$$n_2 = (p - r) + (q + r - 1) = p + q - 1$$

Since the sequence is bounded there are only for finitely many (α, β) such that $\alpha+\beta = p+q+1$ and $E^r_{\alpha,\beta} \neq 0$ for all r. Then for big enough r :

$$E^r_{p+r,q-r+1} = 0 \quad \text{and} \quad E^r_{p-r,q+r-1} = 0, \quad \text{for all } r \gg 0$$

Hence, we can find sufficiently big r such that $0 \longrightarrow E^r_{p,q} \longrightarrow 0$. Hence, $H_{p,q}(E^r) = E^r_{p,q}$. Therefore, since the $r + 1$ - th page is achieved by taking the homology in the r - th page, then

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots$$

Remark 1.4.1. Upshot: If E^r is bounded, then for each pair (p, q), for big enough r:

$$E^r_{p,q} = E^{r+1}_{p,q} = \cdots := E^\infty_{p,q}$$

Exercise 1.4.2. Show that each first quadrant spectral sequence is bounded.

1.4.2 Limit Page Construction

Now we give the following construction. Consider a cohomological spectral sequence $E_r^{p,q}$. Set $\mathbf{B}_0 = 0$ and $\mathbf{Z}_0 := E_0$ so that $E_0 = \mathbf{Z}_0/\mathbf{B}_0$.

1st Step: Set $\mathbf{Z}_1 := \ker\left(E_0 \xrightarrow{d_0} E_0\right)$, and in particular for all (p,q):

$$\mathbf{Z}_1^{p,q} = \ker\left(E_0^{p,q} \xrightarrow{d_0} E_0^{p,q+1}\right)$$

and we define $\mathbf{B}_1$ as $\mathbf{B}_1 =: \operatorname{Im}\left(E_0 \xrightarrow{d_0} E_0\right)$, i.e. for all (p,q) we define :

$$\mathbf{B}_1^{p,q} = \operatorname{Im}\left(E_0^{p,q-1} \xrightarrow{d_0} E_0^{p,q}\right)$$

Notice that $E_1 = H^{p,q}(E_0) = \mathbf{Z}_1/\mathbf{B}_1$.

2nd Step: Take the morphism: $\mathbf{Z}_1 \longrightarrow E_1 = \mathbf{Z}_1/\mathbf{B}_1 \xrightarrow{d_1} E_1$ which at each term (p,q) is defined as follows :

$$\mathbf{Z}_1^{p,q} = \ker(d_0) \longrightarrow E_1^{p,q} = \frac{Z_1^{p,q}}{\mathbf{B}_1^{p,q}} \xrightarrow{d_1} E_1^{p+1,q}$$

We define

$$\mathbf{Z}_2 = \ker\left(\mathbf{Z}_1 \to E_1 \xrightarrow{d_1} E_1\right) \subseteq \mathbf{Z}_1$$

be the elements of $\mathbf{Z}_1$ whose class is killed by d_1. Now let's try carefully to construct $\mathbf{B}_2$. Take the morphism $d_1 : E_1^{p-1,q+1} \to E_1^{p,q}$. For, $\bar{x} \in E_1^{p-1,q+1} = \mathbf{Z}_1^{p-1,q+1}/\mathbf{B}_1^{p-1,q+1}$, we have

$$\bar{x} \in E_1^{p-1,q+1} = \frac{\mathbf{Z}_1^{p-1,q+1}}{\mathbf{B}_1^{p-1,q+1}} \Rightarrow d_1(\bar{x}) \in E_1^{p,q} = \frac{\mathbf{Z}_1^{p,q}}{\mathbf{B}_1^{p,q}} \Rightarrow d_1(\bar{x}) = \overline{y} \in \frac{\mathbf{Z}_1^{p,q}}{\mathbf{B}_1^{p,q}}$$

where $y \in \mathbf{Z}_1^{p,q}$. We define :

$$\mathbf{B}_2^{p,q} = \{y \in \mathbf{Z}_1^{p,q} \mid \overline{y} \in \operatorname{im} d_1\} \subseteq \mathbf{Z}_1^{p,q}$$

Therefore,

$$\frac{\mathbf{B}_2}{\mathbf{B}_1} = \operatorname{Im}\left(E_1 \xrightarrow{d_1} E_1\right)$$

Additionally, by definition we have that

$$E_2^{p,q} = HE_1^{p,q} = \ker d_1^{p,q}/\operatorname{Im} d_1^{p+1,q}.$$

In this step we show that

$$E_2 = HE_1 = \frac{\ker d_1}{\operatorname{Im} d_1} \cong \frac{\mathbf{Z}_2}{\mathbf{B}_2}$$

By definition

$$\mathbf{Z}_2 = \{x \in \mathbf{Z}_1 \mid d_1(\bar{x}) = 0\} \quad \text{and} \quad \mathbf{B}_2 = \{y \in \mathbf{Z}_1 \mid \overline{y} \in \operatorname{Im} d_1\}$$

define a map

$$\varphi : \frac{\mathbf{Z}_2}{\mathbf{B}_2} \longrightarrow \frac{\ker \mathrm{d}_1}{\operatorname{Im} \mathrm{d}_1}, \quad x \mod \mathbf{B}_2 \longmapsto x \mod \operatorname{Im} \mathrm{d}_1$$

this is well defined and isomorphism.

General Case : Now we are ready to describe the general construction. We consider that $\mathbf{B}_r \subseteq \mathbf{Z}_r$, and $E_r \cong \mathbf{Z}_r/\mathbf{B}_r$. Now we define

$$\mathbf{Z}_{r+1} = \ker\left(\mathbf{Z}_r \to E_r = \frac{\mathbf{Z}_r}{\mathbf{B}_r} \xrightarrow{\mathrm{d}_r} E_r\right)$$

and $\mathbf{B}_{r+1}$ such that $\mathbf{B}_r \subseteq \mathbf{B}_{r+1} \subseteq \mathbf{Z}_r$ and is true that :

$$\frac{\mathbf{B}_{r+1}}{\mathbf{B}_r} = \operatorname{Im}\left(\mathbf{Z}_r \to E_r \xrightarrow{\mathrm{d}_r} E_r\right).$$

Thus this construction gives the following chains :

$$0 \subseteq \mathbf{B}_0 \subseteq \mathbf{B}_1 \subseteq \mathbf{B}_2 \subseteq \cdots \subseteq \mathbf{Z}_r \subseteq \mathbf{Z}_{r-1} \subseteq \cdots \subseteq \mathbf{Z}_1 \subseteq \mathbf{Z}_0 = E_0.$$

We also define

$$\mathbf{B}_\infty = \bigcup_i \mathbf{B}_i \quad \text{and} \quad \mathbf{Z}_\infty = \bigcap_i \mathbf{Z}_i$$

Encode the above definitions :

$\mathbf{B}_\infty^{p,q}$ consist the elements of $E_0^{p,q}$ that are image of a differential at some point.

$\mathbf{Z}_\infty^{p,q}$: elements of $E_0^{p,q}$ that are in the kernel of all differentials, so they survive forever.

Definition 1.4.2. Given a cohomological spectral sequence $E = (E_r)$, the E_∞ page is the bigraded module given by

$$E_\infty^{p,q} = \frac{\mathbf{Z}_\infty^{p,q}}{\mathbf{B}_\infty^{p,q}}.$$

Similarly, we define the E^∞ page for a homological spectral sequence.

Lemma 1.4.1. $E_r^{p,q}$ is a cohomological spectral sequence. We have

$$E_{r+1} = E_r \Longleftrightarrow \mathbf{Z}_{r+1} = \mathbf{Z}_r \quad \text{and} \quad \mathbf{B}_{r+1} = \mathbf{B}_r.$$

Proof. If $\mathbf{Z}_{r+1} = \mathbf{Z}_r$ and $\mathbf{B}_{r+1} = \mathbf{B}_r$, then the following quotients are equal :

$$\frac{\mathbf{Z}_{r+1}}{\mathbf{B}_{r+1}} = \frac{\mathbf{Z}_r}{\mathbf{B}_r} \Rightarrow E_{r+1} = E_r.$$

Conversely, we assume $E = E_{r+1} = E_r$. [1] Now since $E_{r+1} = E_r \Rightarrow$

$$\frac{\mathbf{Z}_{r+1}}{\mathbf{B}_{r+1}} = \frac{\mathbf{Z}_r}{\mathbf{B}_r} \iff \frac{\mathbf{Z}_{r+1}/\mathbf{B}_r}{\mathbf{B}_{r+1}/\mathbf{B}_r} = \frac{\mathbf{Z}_r}{\mathbf{B}_r}$$

Now by the fact given at the footnote:

$$\frac{\mathbf{B}_{r+1}}{\mathbf{B}_r} = 0 \Rightarrow \frac{\mathbf{Z}_{r+1}}{\mathbf{B}_r} = \frac{\mathbf{Z}_r}{\mathbf{B}_r} \Rightarrow \mathbf{Z}_{r+1} = \mathbf{Z}_r.$$

□

From the previous lemma we conclude something which was intuitive obvious. If the spectral sequence is eventually constant in terms of pages and the corresponding objects, i.e. $E_r = E_{r+1} = \cdots$, then $E_r = E_{r+1} = \cdots = E_\infty$.

Definition 1.4.3. A spectral sequence $E = (E_r)$ **degenerates** at the n-th page if $\mathrm{d}_r = 0$ for all $r \geq n$

Remark 1.4.2. Hence, for $n \geq r$ $E_n^{p,q} \xrightarrow{\mathrm{d}_n} E_n^{p+r,q-n+1} = 0$. So

$$E_{n+1}^{p,q} = HE_n^{p,q} = \frac{E_n^{p,q}}{0} = E_n^{p,q}.$$

Hence, the spectral sequence stabilizes $E_r = E_{r+1} = E_{r+2} = \cdots$, and from Lemma 1.4.1 we get

$$\mathbf{Z}_r = \mathbf{Z}_{r+1} = \mathbf{Z}_{r+2} = \cdots = \mathbf{Z}_\infty \quad \text{and} \quad \mathbf{B}_r = \mathbf{B}_{r+1} = \cdots = \mathbf{B}_\infty$$

So $E_\infty = \mathbf{Z}_\infty/\mathbf{B}_\infty = \mathbf{Z}_r/\mathbf{B}_r = E_r$

Another more general, but still extreme situation is the following:

Definition 1.4.4. If the r-th page is concentrated in one row or one column, we say that the spectral sequence **collapses** at the r-th page. That obviously, induces that $\mathrm{d}_r = 0$, so E degenerates at the r-th page.

1.4.3 Convergence of a cohomological spectral sequence

Definition 1.4.5. Let $H = (H^n)_{n\in\mathbb{Z}}$ be a graded module. We say that a bounded spectral sequence $E = (E_r)$ converges to H if there is a bounded decreasing filtration $F^\bullet$ for H:

$$E_\infty^{p,q} \cong \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}.$$

We denote by $E_r^{p,q} \Rightarrow H_{p+q}$.

[1] In general, if X/I subquotient of Z, then $I \subseteq X \subseteq Z$ and

$$X/I = Z \iff I = 0 \quad \text{and} \quad X = Z.$$

Remark 1.4.3. Usually we want to understand the objects H^n (for e.g. the cohomology) of some complex, and use a spectral sequence to approximate it via the previous filtration.

Lemma 1.4.2. Let $(E_r^{p,q}, r \geq 0)$ be a spectral sequence such that for all p, q the stable value $E_\infty^{p,q}$ exists. Suppose $E_r^{p,q} \Rightarrow H^{p+q}$.

a. If the spectral sequence is first quadrant then $H^n = 0$ if $n < 0$ and if $n \geq 0$ then H^n has a (decreasing) filtration

$$H^n = F^0 H^n \supseteq \cdots \supseteq F^{n+1} H^n = 0$$

of length $n+1$ where

$$\frac{F^p H^n}{F^{p+1} H^n} \cong E_\infty^{p,n-p} \quad \text{for all } 0 \leq p \leq n.$$

b. Suppose only $E_\infty^{p,\bullet}$ is non zero (for e.g. if E_r collapses to column p). Then $H^n \cong E_\infty^{p,n-p}$.

c. Suppose only $E_\infty^{\bullet,q}$ is non zero. Then $H^n \cong E_\infty^{n-q,q}$.

Proof. Let

$$H^n = F^s H^n \supseteq \cdots \supseteq F^p H^n \supseteq F^{p+1} H^n \supseteq \cdots \supseteq F^t H^n = 0$$

be a filtration for H^n such that

$$\frac{F^p H^n}{F^{p+1} H^n} \cong E_\infty^{p,n-p}.$$

If $n < 0$ then $p < 0$ or $n - p < 0$ so $E_\infty^{p,n-p} = 0$, hence we get :

$$F^p H^n = F^{p+1} H^n \quad \text{for all } p \quad \Rightarrow H^n = 0.$$

If $n \geq 0$, then again $F^p H^n = F^{p+1} H^n$ if $p < 0$ or $p > n+1$. Hence the filtration shortens to

$$H^n = F^0 H^n \supset \cdots \supset F^n H^n \supset F^{n+1} H^n = 0$$

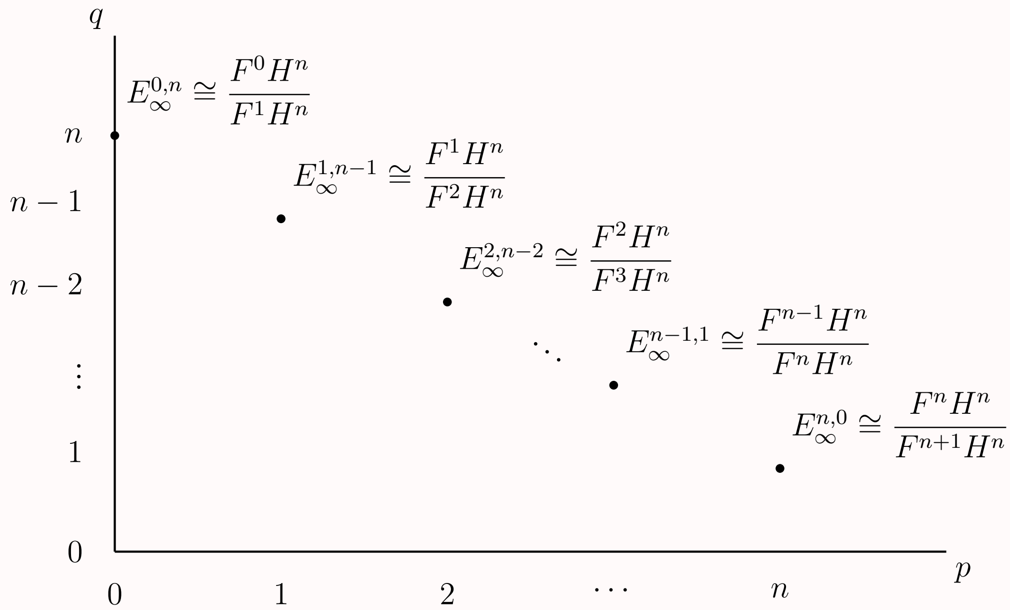
with

$$\frac{F^p H^n}{F^{p+1} H^n} \cong E_\infty^{p,n-p} \quad \text{for all } 0 \leq p \leq n.$$

(2) and (3) are proved similarly. □

Remark 1.4.4. A collapsing spectral sequence always converges to the stable values determined by the nonzero row/column on the collapsing page.

Part (a) of the Lemma 1.4.2 can be visualised as follows :



$$H^n = F^0H^n \supseteq F^1H^n \supseteq \cdots \supseteq F^nH^n \supseteq F^{n+1}H^n = 0$$

1.4.4 Convergence of a homological spectral sequence

Definition 1.4.6. Let $H = (H_n)_{n\in\mathbb{Z}}$ be a graded module. We say that a bounded homological spectral sequence $E = (E^r_{p,q})$ converges to H if there is a bounded increasing filtration $F_\bullet$ for H:

$$0 = F_{-1}H_n \subseteq F_0H_n \subseteq \cdots \subseteq F_nH_n = H_n$$

such that

$$E^\infty_{p,q} \cong \frac{F_pH_{p+q}}{F_{p-1}H_{p+q}}.$$

We denote by $E^r_{p,q} \Rightarrow H_{p+q}$.

Lemma 1.4.3. Let $(E^r_{p,q}, r \geq 0)$ be a homological spectral sequence such that for all p, q the stable value $E^\infty_{p,q}$ exists. Suppose $E^r_{p,q} \Rightarrow H_{p+q}$.

a. If the spectral sequence is first quadrant then $H_n = 0$ if $n < 0$ and if $n \geq 0$ then H_n has an increasing filtration

$$0 = F_{-1}H_n \subseteq F_0H_n \subseteq \cdots \subseteq F_nH_n = H_n$$

of length $n+1$ where

$$\frac{F_pH_n}{F_{p-1}H_n} \cong E^\infty_{p,n-p} \quad \text{for all } 0 \leq p \leq n.$$

b. Suppose only $E^\infty_{p,\bullet}$ is non zero (for e.g. if E^r collapses to column p). Then

$$H_n \cong E^\infty_{p,n-p}.$$

c. Suppose only $E^\infty_{\bullet,q}$ is non zero. Then

$$H_n \cong E^\infty_{n-q,q}.$$

1.4.5 Some Interesting Special Cases of Convergence

We work with a convergent first-quadrant homological spectral sequence

$$E^r_{p,q} \Rightarrow H_{p+q}, \qquad \mathrm{d}^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

Recall that convergence means that for each n there is a finite filtration

$$0 = (H_n)_{-1} \subseteq (H_n)_0 \subseteq \cdots \subseteq (H_n)_n = H_n$$

such that

$$E^\infty_{p,q} \cong (H_{p+q})_p/(H_{p+q})_{p-1}.$$

Exercise 1.4.3. Suppose that $E^2_{p,q} = 0$ unless $p = 0$ or $p = 1$. Then for every $n \geq 0$ there is a short exact sequence

$$0 \longrightarrow E^2_{0,n} \longrightarrow H_n \longrightarrow E^2_{1,n-1} \longrightarrow 0.$$

Proof. Since only the columns $p = 0, 1$ are nonzero on the E^2-page, every differential d^r with $r \geq 2$ must vanish. Indeed:

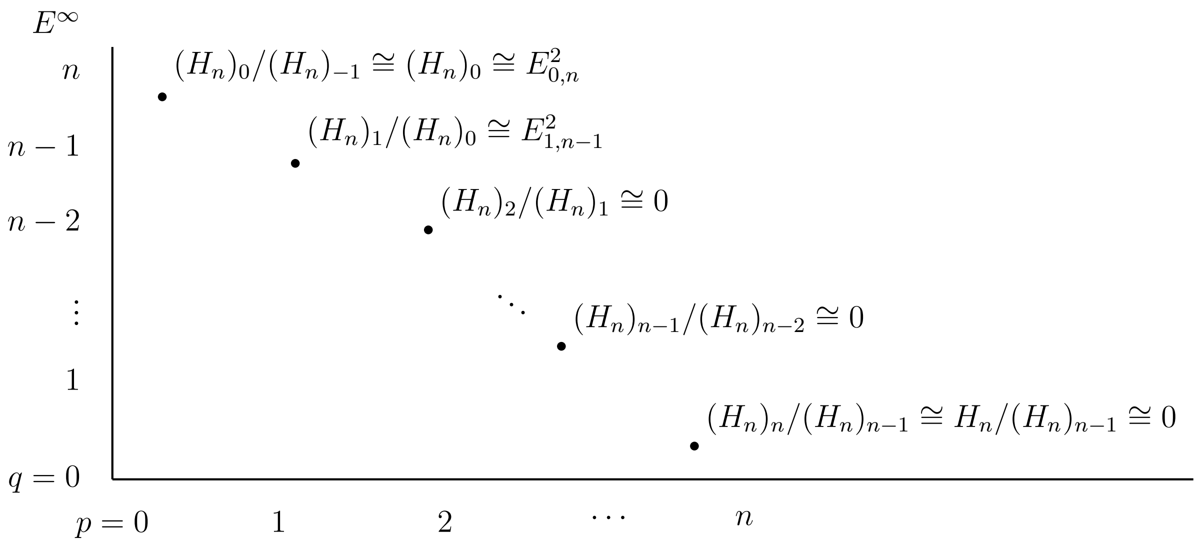
- if $p = 0$ or 1, then the target of

  $$\mathrm{d}^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

  has $p - r < 0$, hence is zero;

- there are no nonzero source terms with $p \geq 2$, so there are no incoming differentials either.

Therefore $E^2_{p,q} = E^\infty_{p,q}$. Now fix n. Since the only possibly nonzero $E^\infty_{p,q}$ with $p + q = n$ are $E^\infty_{0,n}$ and $E^\infty_{1,n-1}$.



The filtration on H_n has the form

$$0 = (H_n)_{-1} \subseteq (H_n)_0 \subseteq (H_n)_1 = H_n,$$

with

$$(H_n)_0 \cong E^\infty_{0,n}, \qquad H_n/(H_n)_0 \cong E^\infty_{1,n-1}.$$

Hence

$$E^2_{0,n} \cong (H_n)_0, \quad E^2_{1,n-1} \cong (H_n)_1/(H_n)_0 \quad \text{and} \quad (H_n)_1 \cong \cdots \cong (H_n)_n = H_n$$

$$0 \longrightarrow E^\infty_{0,n} \longrightarrow H_n \longrightarrow E^\infty_{1,n-1} \longrightarrow 0.$$

Since $E^\infty = E^2$, this becomes

$$0 \longrightarrow E^2_{0,n} \longrightarrow H_n \longrightarrow E^2_{1,n-1} \longrightarrow 0.$$

□

Exercise 1.4.4. Suppose that $E^2_{p,q} = 0$ unless $q = 0$ or 1. Then there is a long exact sequence

$$\cdots \longrightarrow H_{p+1} \longrightarrow E^2_{p+1,0} \xrightarrow{\mathrm{d}^2} E^2_{p-1,1} \longrightarrow H_p \longrightarrow E^2_{p,0} \xrightarrow{\mathrm{d}^2} E^2_{p-2,1} \longrightarrow H_{p-1} \longrightarrow \cdots .$$

Proof. Since only the rows $q = 0, 1$ are nonzero, the only possibly nonzero differential for $r \geq 2$ is

$$\mathrm{d}^2 : E^2_{p,0} \longrightarrow E^2_{p-2,1}.$$

Indeed, for $r \geq 3$ the target row has index $q + r - 1 \geq 2$, hence vanishes. Therefore $E^3 = E^\infty$. Fix n. Since only the terms $E^\infty_{n,0}$ and $E^\infty_{n-1,1}$ may be nonzero on the diagonal $p + q = n$, convergence gives a short exact sequence

$$0 \longrightarrow E^\infty_{n-1,1} \longrightarrow H_n \longrightarrow E^\infty_{n,0} \longrightarrow 0.$$

Now

$$E^{\infty}_{n,0} = E^{3}_{n,0} = \ker\left(\mathrm{d}^2 : E^2_{n,0} \to E^2_{n-2,1}\right),$$

because the $E^3_{n,0}$ is the homology of the E^2 - page at the position $(n,0)$, i.e. the homology of

$$\cdots \to 0 \xrightarrow{\mathrm{d}^2} E^2_{n,0} \xrightarrow{\mathrm{d}^2} E^2_{n-2,1} \to 0 \to \cdots$$

Similarly, we get

$$E^{\infty}_{n-1,1} = \operatorname{coker}\left(\mathrm{d}^2 : E^2_{n+1,0} \to E^2_{n-1,1}\right),$$

because $E^{\infty}_{n-1,1} \cong E^3_{n-1,1}$, and the last object is the homology at the position $(n-1,1)$ of the following sequence

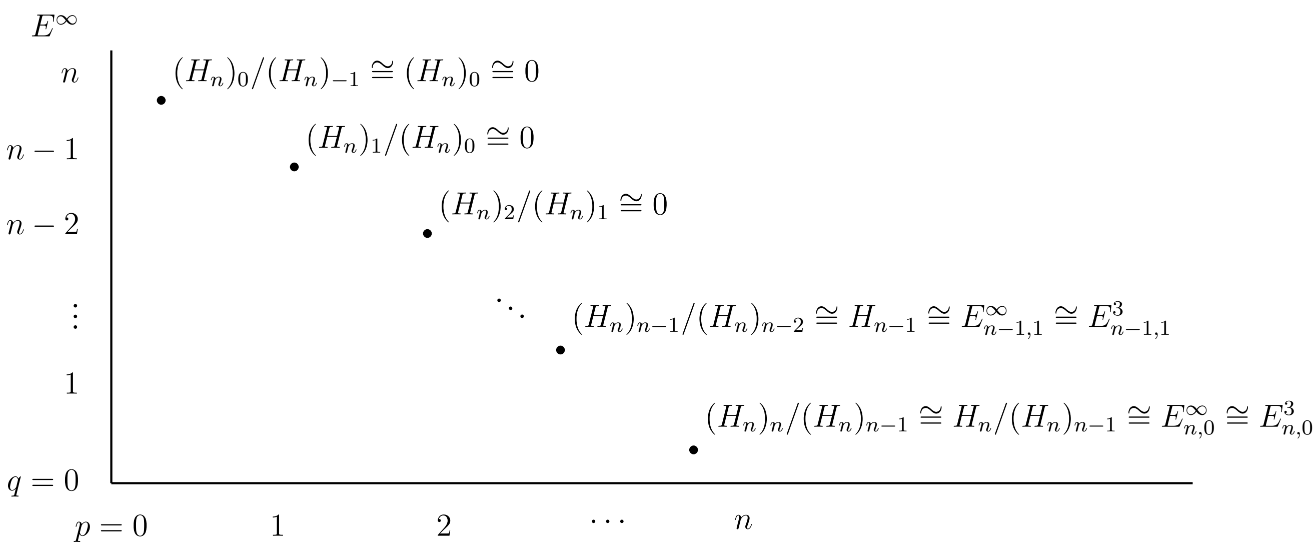
$$\cdots \to E^2_{n+1,0} \xrightarrow{\mathrm{d}^2} E_{n-1,1} \to 0 \to \cdots$$

Thus we have short exact sequences

$$0 \longrightarrow E^{\infty}_{n,0} \longrightarrow E^2_{n,0} \xrightarrow{\mathrm{d}^2} E^2_{n-2,1},$$

and

$$E^2_{n+1,0} \xrightarrow{\mathrm{d}^2} E^2_{n-1,1} \longrightarrow E^{\infty}_{n-1,1} \longrightarrow 0.$$



Combining the previous observations with the following short exact sequence :

$$0 \longrightarrow E^{\infty}_{n-1,1} \longrightarrow H_n \longrightarrow E^{\infty}_{n,0} \longrightarrow 0$$

yields the long exact sequence

$$\cdots \longrightarrow H_{n+1} \longrightarrow E^2_{n+1,0} \xrightarrow{\mathrm{d}^2} E^2_{n-1,1} \longrightarrow H_n \longrightarrow E^2_{n,0} \xrightarrow{\mathrm{d}^2} E^2_{n-2,1} \longrightarrow H_{n-1} \longrightarrow \cdots.$$

□

Exercise 1.4.5 (The five-term exact sequence)**.** Let $E^2_{p,q} \Rightarrow H_{p+q}$ be a convergent first-quadrant homological spectral sequence. Then there is an exact sequence

$$H_2 \longrightarrow E^2_{2,0} \xrightarrow{\mathrm{d}^2} E^2_{0,1} \longrightarrow H_1 \longrightarrow E^2_{1,0} \longrightarrow 0.$$

Proof. We examine the low-degree diagonals.

The group H_1. On the diagonal $p+q=1$, the only possibly nonzero terms are $(1,0)$ and $(0,1)$. Since there is no differential entering or leaving $(1,0)$ (the spectral sequence is of first quandrant) for degree reasons, $E^\infty_{1,0} = E^2_{1,0}$. Also, the only possible differential affecting $(0,1)$ is

$$\mathrm{d}^2 : E^2_{2,0} \longrightarrow E^2_{0,1},$$

so

$$E^\infty_{0,1} = \operatorname{coker}\Big(\mathrm{d}^2 : E^2_{2,0} \to E^2_{0,1}\Big).$$

Therefore, using precisely the same strategy as the previous exercises, convergence yields a short exact sequence

$$0 \longrightarrow E^\infty_{0,1} \longrightarrow H_1 \longrightarrow E^\infty_{1,0} \longrightarrow 0,$$

hence

$$E^2_{2,0} \xrightarrow{\mathrm{d}^2} E^2_{0,1} \longrightarrow H_1 \longrightarrow E^2_{1,0} \longrightarrow 0.$$

Exactness at $E^2_{2,0}$. On the diagonal $p+q=2$, the term $E^\infty_{2,0}$ is the quotient $(H_2)_2/(H_2)_1$. Hence there is a natural surjection $H_2 \twoheadrightarrow E^\infty_{2,0}$. But there are no incoming differentials to $(2,0)$, so by looking the page E^2 we obtain :

$$E^\infty_{2,0} = \ker\Big(\mathrm{d}^2 : E^2_{2,0} \to E^2_{0,1}\Big).$$

Composing the natural map

$$H_2 \twoheadrightarrow E^\infty_{2,0} \hookrightarrow E^2_{2,0}$$

shows that the image of $H_2 \to E^2_{2,0}$ is exactly

$$\ker\Big(d^2 : E^2_{2,0} \to E^2_{0,1}\Big).$$

Therefore the sequence

$$H_2 \longrightarrow E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \longrightarrow H_1 \longrightarrow E^2_{1,0} \longrightarrow 0$$

is exact. $\square$

Exercise 1.4.6 (Edge morphisms)**.** Let $\{E^r_{p,q}\}$ be a homological spectral sequence. For every $r \geq 2$ (and $r \geq a$ if the spectral sequence starts on the E^a-page), there are natural maps

$$E^r_{0,n} \longrightarrow E^\infty_{0,n} \qquad \text{and} \qquad E^\infty_{n,0} \longrightarrow E^r_{n,0}.$$

More precisely, the first map is a natural surjection and the second is a natural injection:

$$E^r_{0,n} \twoheadrightarrow E^\infty_{0,n}, \qquad E^\infty_{n,0} \hookrightarrow E^r_{n,0}.$$

If in addition the spectral sequence converges to H_*, then $E^\infty_{0,n} \cong (H_n)_0 \subseteq H_n$, and $E^\infty_{n,0} \cong H_n/(H_n)_{n-1}$. Hence we obtain natural maps

$$E^r_{0,n} \longrightarrow H_n \quad \text{and} \quad H_n \longrightarrow E^r_{n,0},$$

namely

$$E^r_{0,n} \twoheadrightarrow E^\infty_{0,n} \hookrightarrow H_n$$

and

$$H_n \twoheadrightarrow E^\infty_{n,0} \hookrightarrow E^r_{n,0}.$$

These are called the **edge morphisms**.

Chapter 2

Examples of Spectral Sequences

2.1 Double Complexes

2.1.1 Double Complexes

Definition 2.1.1. Let $\mathscr{A}$ be an additive category. A **double co-complex** is a collection of objects $(A^{p,q})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ along with morphisms

$$\mathrm{d}_h^{p,q} : A^{p,q} \longrightarrow A^{p+1,q}$$

and

$$\mathrm{d}_v^{p,q} : A^{p,q} \longrightarrow A^{p,q+1}$$

such that for all $p, q \in \mathbb{Z}$:

1. $\mathrm{d}_h^{p+1,q} \circ \mathrm{d}_h^{p,q} = 0$,
2. $\mathrm{d}_v^{p,q+1} \circ \mathrm{d}_v^{p,q} = 0$,
3. the square

$$\begin{array}{ccc} A^{p,q} & \xrightarrow{\mathrm{d}_h^{p,q}} & A^{p+1,q} \\ {\scriptstyle \mathrm{d}_v^{p,q}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{d}_v^{p+1,q}} \\ A^{p,q+1} & \xrightarrow[\mathrm{d}_h^{p,q+1}]{} & A^{p+1,q+1} \end{array}$$

   commutes (sometimes the convention is that this square anti-commutes, i.e.

$$\mathrm{d}_v^{p+1,q}\mathrm{d}_h^{p,q} + \mathrm{d}_h^{p,q+1}\mathrm{d}_v^{p,q} = 0).$$

   For all $p, q \in \mathbb{Z}$, $A^{p,\bullet}$ and $A^{\bullet,q}$ are co-complexes

$$\mathrm{d}_h^{p,\bullet} : A^{p,\bullet} \longrightarrow A^{p+1,\bullet}$$

and

$$\mathrm{d}_v^{\bullet,q} : A^{\bullet,q} \longrightarrow A^{\bullet,q+1}$$

are maps of co-complexes.

Remark 2.1.1. $A^{\bullet,\bullet}$ can be viewed as a co-complex of co-complexes in one of two ways, i.e. as an object of $C(C(\mathscr{A}))$.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \mathrm{d}_v} & & \uparrow{\scriptstyle \mathrm{d}_v} & & \uparrow{\scriptstyle \mathrm{d}_v} & & \\
\cdots & \longrightarrow & A^{p-1,q+1} & \xrightarrow{\mathrm{d}_h} & A^{p,q+1} & \xrightarrow{\mathrm{d}_h} & A^{p+1,q+1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \mathrm{d}_v} & & \uparrow{\scriptstyle \mathrm{d}_v} & & \uparrow{\scriptstyle \mathrm{d}_v} & & \\
\cdots & \longrightarrow & A^{p-1,q} & \xrightarrow{\mathrm{d}_h} & A^{p,q} & \xrightarrow{\mathrm{d}_h} & A^{p+1,q} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \mathrm{d}_v} & & \uparrow{\scriptstyle \mathrm{d}_v} & & \uparrow{\scriptstyle \mathrm{d}_v} & & \\
\cdots & \longrightarrow & A^{p-1,q-1} & \xrightarrow{\mathrm{d}_h} & A^{p,q-1} & \xrightarrow{\mathrm{d}_h} & A^{p+1,q-1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

One similarly defines a double complex using homological indexing and reversing the arrows in the above grid.

Remark 2.1.2. Both double cocomplexes and double complexes are usually just called double complexes. We will use $A^{\bullet\bullet}$ to denote a double cocomplex and $A_{\bullet\bullet}$ to denote a double complex.

Definition 2.1.2. A **first quadrant double complex** $A^{\bullet\bullet}$ is one where

$$A^{p,q} = 0 \quad \text{if } p < 0 \text{ or } q < 0.$$

There are similar double complexes for the other quadrants. An **upper half plane** double complex $A^{\bullet\bullet}$ is one where

$$A^{p,q} = 0 \quad \text{for all } q < 0.$$

One similarly defines lower/left/right half plane double complexes.

2.1.2 The total complex of a double complex

Assume $\mathscr{A} = \mathrm{Mod}_R$ for simplicity. Let $A^{\bullet\bullet}$ be a double complex of R-modules. For $n \in \mathbb{Z}$, the terms $A^{p,q}$ such that $p + q = n$ lie on the diagonal line $x + y = n$. The **total complex** of $A^{\bullet\bullet}$, denoted $\mathrm{Tot}(A^{\bullet\bullet})$, is an object of $C(\mathrm{Mod}_R)$ defined as follows:

$$\mathrm{Tot}(A^{\bullet\bullet})^n := \bigoplus_{p+q=n} A^{p,q}.$$

The differential of $\mathrm{Tot}\left(A^{\bullet\bullet}\right)$ defined as follows :

$$\mathrm{d}^n : \bigoplus_{p+q=n} A^{p,q} \longrightarrow \bigoplus_{s+t=n+1} A^{s,t}$$

is uniquely determined by specifying the maps

$$A^{p,q} \longrightarrow \bigoplus_{s+t=n+1} A^{s,t}, \qquad x \longmapsto (y_{s,t})_{s+t=n+1}$$

where

$$y_{s,t} = \begin{cases} \mathrm{d}_h^{p,q}(x) & (s,t) = (p+1, q), \\ \\ (-1)^p \cdot \mathrm{d}_v^{p,q}(x) & (s,t) = (p, q+1), \\ \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2.1.1. Show that $\mathrm{d}^{n+1} \circ \mathrm{d}^n = 0$, i.e. $\mathrm{Tot}(A^{\bullet\bullet})$ is indeed a complex.

The sign $(-1)^p$ in the definition of d^n makes the definition of $\mathrm{Tot}(A^{\bullet\bullet})$ not symmetric in p and q. The differential d^n is commonly denoted as

$$\bigoplus_{p+q=n} \left(\mathrm{d}_h^{p,q} + (-1)^p \mathrm{d}_v^{p,q}\right).$$

Remark 2.1.3. If $A^{\bullet\bullet}$ is a first quadrant double complex, then for all $n \in \mathbb{Z}$ there are only finitely many $A^{p,q} \neq 0$ with $p + q = n$.

In particular, the terms of $\mathrm{Tot}(A^{\bullet\bullet})$ are finite coproducts.

Example 2.1.1 (Tensor product of complexes)**.** The tensor product of $A^\bullet$ and $B^\bullet$ is defined as

$$\mathrm{Tot}(A^\bullet \otimes_R B^\bullet).$$

Explicitly,

$$\mathrm{Tot}(A^\bullet \otimes_R B^\bullet)^n = \bigoplus_{p+q=n} A^p \otimes_R B^q,$$

and the differential is

$$\mathrm{d}^n = \bigoplus_{p+q=n} \left(\mathrm{d}_h^{p,q} + (-1)^p \mathrm{d}_v^{p,q}\right) = \bigoplus_{p+q=n} \left(\mathrm{d}_A^p \otimes \mathrm{id} + (-1)^p \mathrm{id} \otimes \mathrm{d}_B^q\right).$$

Sometimes $\mathrm{Tot}(A^\bullet \otimes_R B^\bullet)$ is denoted simply by $A^\bullet \otimes_R B^\bullet$.

Exercise 2.1.2. View a complex $A^\bullet$ as a double complex $B^{\bullet\bullet}$ concentrated in the p-th column, i.e.

$$B^{i,\bullet} = \begin{cases} 0 & \text{if } i \neq p, \\ \\ A^\bullet & \text{if } i = p. \end{cases}$$

All horizontal maps are zero. Show that $\text{Tot}(B^{\bullet\bullet}) \cong A^\bullet[-p]$.

Exercise 2.1.3. Let $\varphi^\bullet : A^\bullet \to B^\bullet$ be a morphism of complexes. Recover the mapping cone C_φ as the total complex of an appropriately chosen double complex.

Exercise 2.1.4. Let $A^{\bullet\bullet}$ be a double complex. Define its transpose $A_T^{\bullet\bullet}$ by switching rows and columns: $A_T^{p,q} = A^{q,p}$. Show that

$$\text{Tot}(A_T^{\bullet\bullet}) \cong \text{Tot}(A^{\bullet\bullet}).$$

2.1.3 The product total complex

Given $A^{\bullet\bullet}$, the product total complex is defined by

$$\text{Tot}^{\Pi}(A^{\bullet\bullet})^n = \prod_{p+q=n} A^{p,q},$$

and the differential

$$\mathrm{d}^n : \text{Tot}^{\Pi}(A^{\bullet\bullet})^n \longrightarrow \text{Tot}^{\Pi}(A^{\bullet\bullet})^{n+1}$$

is given by

$$\prod_{p+q=n} \left(\mathrm{d}_h^{p,q} + (-1)^p \mathrm{d}_v^{p,q}\right).$$

Thus $\text{Tot}(A^{\bullet\bullet})$ is a subcomplex of $\text{Tot}^{\Pi}(A^{\bullet\bullet})$.

Remark 2.1.4. Both $\text{Tot}(A^{\bullet\bullet})$ and $\text{Tot}^{\Pi}(A^{\bullet\bullet})$ can be defined for double complexes in any abelian category $\mathscr{A}$ that has countable products and coproducts.

Definition 2.1.3. A **morphism** of double complexes $f^{\bullet\bullet} : A^{\bullet\bullet} \longrightarrow B^{\bullet\bullet}$ induces morphisms

$$\text{Tot}(f^{\bullet\bullet}) : \text{Tot}(A^{\bullet\bullet}) \longrightarrow \text{Tot}(B^{\bullet\bullet})$$

and

$$\text{Tot}^{\Pi}(f^{\bullet\bullet}) : \text{Tot}^{\Pi}(A^{\bullet\bullet}) \longrightarrow \text{Tot}^{\Pi}(B^{\bullet\bullet}),$$

given respectively by

$$\bigoplus_{p+q=n} f^{p,q} \quad \text{and} \quad \prod_{p+q=n} f^{p,q}.$$

2.2 The spectral sequence of a filtered complex

Now we are ready to show the first important example of a spectral sequence. This is the spectral sequence which arises from a filtered complex. For this construction, we use homological notation. So we start with a filtered complex $(C_\bullet, \mathrm{d}, F)$. We define

$$E^0_{p,q} := \frac{F^pC_{p+q}}{F^{p-1}C_{p+q}}.$$

The differentials $\mathrm{d}_0 : E^0_{p,q} \to E^0_{p,q-1}$ are induced by the differential of $(C_\bullet, \mathrm{d}, F)$, i.e.

$$\tilde{d}\colon \frac{F^pC_{p+q}}{F^{p-1}C_{p+q}} \longrightarrow \frac{F^pC_{p+q-1}}{F^{p-1}C_{p+q-1}}.$$

[1] From the definition of the homological spectral sequence, we are forced to define the first page as the homology of the 0-page. So we define

$$E^1_{p,q} := H_{p+q}\left(\frac{F^pC_\bullet}{F^{p-1}C_\bullet}\right).$$

In simpler words, $E^1_{p,q}$ is the homology induced by

$$\frac{F^pC_{p+q+1}}{F^{p-1}C_{p+q+1}} \xrightarrow{\mathrm{d}_0} \frac{F^pC_{p+q}}{F^{p-1}C_{p+q}} \xrightarrow{\mathrm{d}_0} \frac{F^pC_{p+q-1}}{F^{p-1}C_{p+q-1}}.$$

We also notice that

$$H_{p+q}\left(\frac{F^pC_\bullet}{F^{p-1}C_\bullet}\right) = \frac{\ker(\mathrm{d}_0\colon E^0_{p,q} \to E^0_{p,q-1})}{\mathrm{Im}(\mathrm{d}_0\colon E^0_{p,q+1} \to E^0_{p,q})}.$$

Let us rewrite $E^1_{p,q}$ using the definitions in a more explicit way that motivates the definition of the r - th page. We have

$$E^1_{p,q} = \frac{\{x \in F^pC_{p+q} \mid \mathrm{d}(x) \in F^{p-1}C_{p+q-1}\}}{F^{p-1}C_{p+q} + \mathrm{d}(F^pC_{p+q+1})}.$$

For all $\bar{x} \in \ker \mathrm{d}_0$, we have $\overline{\mathrm{d}(x)} = 0$, so $\mathrm{d}(x) \in F^{p-1}C_{p+q-1}$. Thus the "numerator" of the right module consists exactly of all possible representatives of those classes. Since $E^1_{p,q} = \ker\tilde{\mathrm{d}}/\,\mathrm{Im}\,\tilde{\mathrm{d}}$, we must also quotient by $F^{p-1}C_{p+q}$ "together" with the image of $\mathrm{d} : F^pC_{p+q+1} \to F^pC_{p+q}$, meaning that we quotient with their sum. Hence the denominator is

$$F^{p-1}C_{p+q} + \mathrm{d}(F^pC_{p+q+1}).$$

Now we have to define the differentials on the first page. We define the differential $\mathrm{d}_1 : E^1_{p,q} \to E^1_{p-1,q}$ following the next procedure : If $\bar{x} \in E^1_{p,q}$, then $x \in F^pC_{p+q}$ and $\mathrm{d}(x) \in F^{p-1}C_{p+q-1}$. So there exists $y \in F^{p-1}C_{p+q-1}$ such that $y = \mathrm{d}(x)$. Then

$$\bar{y} = \mathrm{d}(x) \Rightarrow \mathrm{d}(\bar{y}) = \overline{\mathrm{d}(y)} = \overline{\mathrm{d}^2(x)} = 0 \Rightarrow \mathrm{d}(y) \in F^{p-2}C_{p+q-2}.$$

[1] Even in the homological case, the filtration is increasing.

Hence $\bar{y}$ defines an element in $E^1_{p-1,q}$. This construction generalizes inductively. For the r-th page we define

$$E^r_{p,q} := \frac{\{x \in F^p C_{p+q} \mid \mathrm{d}(x) \in F^{p-r} C_{p+q-1}\}}{F^{p-1} C_{p+q} + \mathrm{d}(F^{p+r-1} C_{p+q+1})}. \tag{2.1}$$

and the differentials are give by

$$\mathrm{d}_r : E^r_{p,q} \to E^r_{p-r,q+r-1}, \quad \mathrm{d}_r(\bar{x}) = \overline{\mathrm{d}(x)} \tag{2.2}$$

It is well-defined and satisfies $\mathrm{d}_r \circ \mathrm{d}_r = 0$, and $E^{r+1}_{p,q} \cong H_{p,q}(E^r)$.

Theorem 2.2.1. Let $(C_\bullet, \mathrm{d}, F)$ be a filtered complex. Then 2.1 defines a spectral sequence, with the differentials given in 2.2, satisfying $\mathrm{d}^r \circ \mathrm{d}^r = 0$ and $E^{r+1}_{p,q} \cong H_{p,q}(E^r)$. Thus this gives a spectral sequence with

$$E^1_{p,q} = H_{p+q}\left(\frac{F^p C_\bullet}{F^{p-1} C_\bullet}\right).$$

If the filtration is bounded, then this spectral sequence converges to

$$E^\infty_{p,q} = \frac{F^p H_{p+q}(C_\bullet)}{F^{p-1} H_{p+q}(C_\bullet)}.$$

Proof. Practically, we have already proved the theorem, except for the last part. For the last part, see [1], Chapter 12. □

2.3 Spectral Sequence of a double complex

2.3.1 The first filtration

As in the case of filtered complexes, we adopt the cohomological convention in this set-up. Let $C^{\bullet,\bullet} = (C^{p,q})_{p,q\in\mathbb{Z}}$ be a double complex. We consider the total complex $\mathrm{Tot}(C^{\bullet,\bullet})^n = \bigoplus_{p+q=n} C^{p,q}$, with differential $\mathrm{d} = \mathrm{d}_h + (-1)^p \mathrm{d}_v$.

Definition 2.3.1. The **first (decreasing) filtration** of $\mathrm{Tot}(C^{\bullet,\bullet})$ is defined by

$${}^1F^p \,\mathrm{Tot}(C^{\bullet,\bullet})^n = \bigoplus_{i\geq p} C^{i,n-i}.$$

The filtered complex $(\mathrm{Tot}(C^{\bullet,\bullet}), \mathrm{d}, {}^1F)$ gives a spectral sequence denoted by ${}^1E$. In order to simplify the notation, we write $E^{p,q}_r$ instead of ${}^1E^{p,q}_r$ for the spectral sequence associated with the first filtration. Our spectral sequence starts with

$$E^{p,q}_0 = \frac{{}^1F^p \,\mathrm{Tot}(C^{\bullet,\bullet})^{p+q}}{{}^1F^{p+1} \,\mathrm{Tot}(C^{\bullet,\bullet})^{p+q}} = \frac{\bigoplus_{i\geq p} C^{i,p+q-i}}{\bigoplus_{i\geq p+1} C^{i,p+q-i}} \cong C^{p,q}.$$

Moreover, for the differential $\mathrm{d}_0 : E_0^{p,q} \longrightarrow E_0^{p,q+1}$, it turns out that

$$\mathrm{d}_0 = (-1)^p \mathrm{d}_v : C^{p,q} \longrightarrow C^{p,q+1}.$$

Indeed, the differential on $\mathrm{Tot}(C^{\bullet,\bullet})$ is

$$\mathrm{d} : \mathrm{Tot}(C^{\bullet,\bullet})^n \longrightarrow \mathrm{Tot}(C^{\bullet,\bullet})^{n+1}, \quad \mathrm{d} = \bigoplus_{p+q=n} (\mathrm{d}_h^{p,q} + (-1)^p \mathrm{d}_v^{p,q}).$$

The horizontal differential increases the filtration index, hence it vanishes on the associated graded object. Therefore, only the vertical differential remains on the 0-page. So the 0-page looks like:

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\uparrow{\scriptstyle (-1)^{p-1}\mathrm{d}_v} & \uparrow{\scriptstyle (-1)^{p}\mathrm{d}_v} & \uparrow{\scriptstyle (-1)^{p+1}\mathrm{d}_v} \\
C^{p-1,q+1} & C^{p,q+1} & C^{p+1,q+1} \\
\uparrow{\scriptstyle (-1)^{p-1}\mathrm{d}_v} & \uparrow{\scriptstyle (-1)^{p}\mathrm{d}_v} & \uparrow{\scriptstyle (-1)^{p+1}\mathrm{d}_v} \\
C^{p-1,q} & C^{p,q} & C^{p+1,q} \\
\uparrow{\scriptstyle (-1)^{p-1}\mathrm{d}_v} & \uparrow{\scriptstyle (-1)^{p}\mathrm{d}_v} & \uparrow{\scriptstyle (-1)^{p+1}\mathrm{d}_v} \\
C^{p-1,q-1} & C^{p,q-1} & C^{p+1,q-1} \\
\uparrow & \uparrow & \uparrow \\
\vdots & \vdots & \vdots
\end{array}$$

About the page 1, we get

$${}^{I}E_1^{p,q} = H^q(C^{p,\bullet}, \mathrm{d}_v).$$

The sign $(-1)^p$ does not change the cohomology of the vertical complexes. Thus, even though $\mathrm{d}_0 = (-1)^p \mathrm{d}_v$ we still have $E_1^{p,q} = H^q(C^{p,\bullet}, \mathrm{d}_v)$, Now notice that there is a complex morphism $C^{p,\bullet} \xrightarrow{\mathrm{d}_h} C^{p+1,\bullet}$ looking at the horizontal differential which induces a morphism on cohomology

$$H^q(\mathrm{d}_h) : H^q(C^{p,\bullet}) \longrightarrow H^q(C^{p+1,\bullet}).$$

Therefore, the first differential is $\mathrm{d}_1 : E_1^{p,q} \to E_1^{p+1,q}$, and it coincides with the map induced by d_h on vertical cohomology. Hence $\mathrm{d}_1 = H^q(\mathrm{d}_h)$. So we conclude that

$$E_2^{p,q} = H^p\big(H^q(C^{\bullet,\bullet}, \mathrm{d}_v), \mathrm{d}_h\big).$$

In order to simplify the notation we write $H_h^p H_v^q(C_\bullet)$ instead of $H^p\big(H^q(C^{\bullet,\bullet}, \mathrm{d}_v), \mathrm{d}_h\big)$. Finally, if $E_2^{p,q}$ is bounded, for example if the double complex is first quadrant, then by Theorem 2.2.1 we get :

$$E_2^{p,q} = H_h^p H_v^q(C_\bullet) \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet})).$$

Theorem 2.3.1. Let $C^{\bullet,\bullet}$ be a double complex. The first filtration ${}^1F^p\,\mathrm{Tot}(C^{\bullet,\bullet})^n = \bigoplus_{i\geq p} C^{i,n-i}$ induces a spectral sequence ${}^1E$ such that

$$ {}^1E_1^{p,q} = H^q(C^{p,\bullet}, \mathrm{d}_v) \quad \text{and} \quad {}^1E_2^{p,q} = H_h^p H_v^q(C_\bullet) $$

If the filtration is bounded, then this spectral sequence converges to $H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))$.

2.3.2 The second filtration

Definition 2.3.2. The **second (decreasing) filtration** of $\mathrm{Tot}(C^{\bullet,\bullet})$ is defined by

$$ {}^2F^q\,\mathrm{Tot}(C^{\bullet,\bullet})^n = \bigoplus_{j\geq q} C^{n-j,j}. $$

The filtered complex $(\mathrm{Tot}(C^{\bullet,\bullet}), \mathrm{d}, {}^2F)$ gives a spectral sequence denoted by ${}^2E$. In order to simplify the notation, we write $E_r^{p,q}$ instead of ${}^2E_r^{p,q}$ for the spectral sequence associated with the second filtration. Our spectral sequence starts with

$$ E_0^{p,q} = \frac{{}^2F^q\,\mathrm{Tot}(C^{\bullet,\bullet})^{p+q}}{{}^2F^{q+1}\,\mathrm{Tot}(C^{\bullet,\bullet})^{p+q}} = \frac{\bigoplus\limits_{j\geq q} C^{p+q-j,j}}{\bigoplus\limits_{j\geq q+1} C^{p+q-j,j}} \cong C^{p,q}. $$

Moreover, for the differential $\mathrm{d}_0 : E_0^{p,q} \longrightarrow E_0^{p+1,q}$, it turns out that $\mathrm{d}_0 = \mathrm{d}_h : C^{p,q} \longrightarrow C^{p+1,q}$. Indeed, the differential on $\mathrm{Tot}(C^{\bullet,\bullet})$ is

$$ \mathrm{d} : \mathrm{Tot}(C^{\bullet,\bullet})^n \longrightarrow \mathrm{Tot}(C^{\bullet,\bullet})^{n+1}, \quad \mathrm{d} = \bigoplus_{p+q=n} \left(\mathrm{d}_h^{p,q} + (-1)^p \mathrm{d}_v^{p,q}\right). $$

The vertical differential increases the filtration index, hence it vanishes on the associated graded object. Therefore, only the horizontal differential remains on the 0-page. So the 0-page looks like:

$$ \begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ C^{p-1,q+1} & \xrightarrow{\mathrm{d}_h} & C^{p,q+1} & \xrightarrow{\mathrm{d}_h} & C^{p+1,q+1} \\ & & & & \\ C^{p-1,q} & \xrightarrow{\mathrm{d}_h} & C^{p,q} & \xrightarrow{\mathrm{d}_h} & C^{p+1,q} \\ & & & & \\ C^{p-1,q-1} & \xrightarrow{\mathrm{d}_h} & C^{p,q-1} & \xrightarrow{\mathrm{d}_h} & C^{p+1,q-1} \\ \vdots & & \vdots & & \vdots \end{array} $$

About the page 1, we get ${}^2E_1^{p,q} = H^p(C^{\bullet,q}, \mathrm{d}_h)$. Now notice that there is a complex morphism $C^{\bullet,q} \xrightarrow{(-1)^p \mathrm{d}_v} C^{\bullet,q+1}$ looking at the vertical differential, which induces a morphism on cohomology

$$H^p((-1)^p \mathrm{d}_v) : H^p(C^{\bullet,q}) \longrightarrow H^p(C^{\bullet,q+1}).$$

Therefore, the first differential is

$$\mathrm{d}_1 : E_1^{p,q} \to E_1^{p,q+1},$$

and it coincides with the map induced by $(-1)^p \mathrm{d}_v$ on horizontal cohomology. The sign does not affect the resulting cohomology groups, so we conclude that

$$E_2^{p,q} = H^q\big(H^p(C^{\bullet,\bullet}, \mathrm{d}_h), \mathrm{d}_v\big).$$

In order to simplify the notation we write $H_v^q H_h^p(C_\bullet)$ instead of $H^q\big(H^p(C^{\bullet,\bullet}, \mathrm{d}_h), \mathrm{d}_v\big)$. Finally, if $E_2^{p,q}$ is bounded, for example if the double complex is first quadrant, then by Theorem 2.2.1 we get:

$$E_2^{p,q} = H_v^q H_h^p(C_\bullet) \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))\,.$$

Theorem 2.3.2. Let $C^{\bullet,\bullet}$ be a double complex. The second filtration

$${}^2F^q \,\mathrm{Tot}(C^{\bullet,\bullet})^n = \bigoplus_{j \geq q} C^{n-j,j}$$

induces a spectral sequence ${}^2E$ such that

$${}^2E_1^{p,q} = H^p(C^{\bullet,q}, \mathrm{d}_h) \quad \text{and} \quad {}^2E_2^{p,q} = H_v^q H_h^p(C_\bullet).$$

If the filtration is bounded, then this spectral sequence converges to

$$H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))\,.$$

2.4 The Grothendieck Spectral Sequence

2.4.1 Fully Injective Resolutions

Definition 2.4.1. Let $C^\bullet = (C^p)_{p\in\mathbb{Z}}$ be a cochain complex, and suppose that $C^p = 0$ for $p < 0$ for simplicity. We define an **injective resolution** of $C^\bullet$ to be a resolution

$$0 \longrightarrow C^\bullet \longrightarrow I^{\bullet,\bullet},$$

written briefly as $0 \longrightarrow C \longrightarrow I_C^\bullet$, such that $I^{\bullet,\bullet} = (I^{p,q})_{p,q\geq 0}$ is a double complex in $\mathrm{Ch}^{\geq 0}(\mathrm{Ch}^{\geq 0}(\mathscr{A}))$ (first quadrant), satisfying:

- each $I^{p,q}$ is an injective object,

- for all q, the complex

$$0 \longrightarrow C^q \longrightarrow I^{0,q} \xrightarrow{\mathrm{d}_h^{0,q}} I^{1,q} \xrightarrow{\mathrm{d}_h^{1,q}} I^{2,q} \longrightarrow \cdots$$

  is an injective resolution of C^q.

This can be visualized as an injective resolution of a complex by the following diagram:

$$
\begin{array}{ccccccccccc}
 & & \uparrow{\scriptstyle \mathrm{d}^3} & & \uparrow{\scriptstyle \mathrm{d}_v^{0,3}} & & \uparrow{\scriptstyle \mathrm{d}_v^{1,3}} & & \uparrow{\scriptstyle \mathrm{d}_v^{2,3}} & & \\
0 & \longrightarrow & C^3 & \longrightarrow & I^{0,3} & \xrightarrow{\mathrm{d}_h^{0,3}} & I^{1,3} & \xrightarrow{\mathrm{d}_h^{1,3}} & I^{2,3} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \mathrm{d}^2} & & \uparrow{\scriptstyle \mathrm{d}_v^{0,2}} & & \uparrow{\scriptstyle \mathrm{d}_v^{1,2}} & & \uparrow{\scriptstyle \mathrm{d}_v^{2,2}} & & \\
0 & \longrightarrow & C^2 & \longrightarrow & I^{0,2} & \xrightarrow{\mathrm{d}_h^{0,2}} & I^{1,2} & \xrightarrow{\mathrm{d}_h^{1,2}} & I^{2,2} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \mathrm{d}^1} & & \uparrow{\scriptstyle \mathrm{d}_v^{0,1}} & & \uparrow{\scriptstyle \mathrm{d}_v^{1,1}} & & \uparrow{\scriptstyle \mathrm{d}_v^{2,1}} & & \\
0 & \longrightarrow & C^1 & \longrightarrow & I^{0,1} & \xrightarrow{\mathrm{d}_h^{0,1}} & I^{1,1} & \xrightarrow{\mathrm{d}_h^{1,1}} & I^{2,1} & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \mathrm{d}^0} & & \uparrow{\scriptstyle \mathrm{d}_v^{0,0}} & & \uparrow{\scriptstyle \mathrm{d}_v^{1,0}} & & \uparrow{\scriptstyle \mathrm{d}_v^{2,0}} & & \\
0 & \longrightarrow & C^0 & \longrightarrow & I^{0,0} & \xrightarrow{\mathrm{d}_h^{0,0}} & I^{1,0} & \xrightarrow{\mathrm{d}_h^{1,0}} & I^{2,0} & \longrightarrow & \cdots
\end{array}
$$

Remark 2.4.1. Given an injective resolution as in the definition above, we obtain complexes:

$$0 \longrightarrow \mathbf{Z}^q(C^\bullet) \longrightarrow \mathbf{Z}^{0,q} \to \mathbf{Z}^{1,q} \to \cdots$$

$$0 \longrightarrow \mathbf{B}^q(C^\bullet) \longrightarrow \mathbf{B}^{0,q} \to \mathbf{B}^{1,q} \to \cdots$$

$$0 \longrightarrow H^q(C^\bullet) \longrightarrow H^{0,q} \to H^{1,q} \to \cdots$$

where

$$\mathbf{Z}^{p,q} := \ker(\mathrm{d}_v^{p,q}), \qquad \mathbf{B}^{p,q} := \mathrm{Im}(\mathrm{d}_v^{p,q-1}), \qquad H^{p,q} := \frac{\mathbf{Z}^{p,q}}{\mathbf{B}^{p,q}}.$$

We indicate how the complex $0 \to \mathbf{Z}^q(C^\bullet) \to \mathbf{Z}^{\bullet,q}$ is constructed, and similarly, using the universal property of cokernels (or images), we obtain induced morphisms on $\mathbf{B}^{\bullet,q}$ and hence on the quotients

$$H^{\bullet,q} = \frac{\mathbf{Z}^{\bullet,q}}{\mathbf{B}^{\bullet,q}}.$$

We have the following diagram:

$$
\begin{array}{ccccc}
\mathbf{Z}^q(C^\bullet) & \hookrightarrow & C^q & \xrightarrow{\mathrm{d}^q} & C^{q+1} \\
 & & \downarrow & & \downarrow \\
\mathbf{Z}^{0,q} & \hookrightarrow & I^{0,q} & \xrightarrow{\mathrm{d}_h^{0,q}} & I^{0,q+1}
\end{array}
$$

From the commutativity of the diagram, we obtain that the composition

$$\mathbf{Z}^q(C^\bullet) \to C^q \to I^{0,q+1}$$

is zero. Hence, by the universal property of the kernel, we obtain an induced morphism making the following diagram commute:

$$
\begin{array}{ccccc}
\mathbf{Z}^q(C^\bullet) & \hookrightarrow & C^q & \xrightarrow{\mathrm{d}^q} & C^{q+1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Z}^{0,q} & \hookrightarrow & I^{0,q} & \xrightarrow{\mathrm{d}_h^{0,q}} & I^{0,q+1}
\end{array}
\tag{2.3}
$$

We can easily see that the induced map on the kernels is a monomorphism. Similarly, we construct the remaining differentials.

Warning !

In general, it is not true that the induced complexes are injective resolutions, nor even resolutions. This observation leads naturally to the next definition.

Definition 2.4.2. We say that the resolution

$$0 \longrightarrow C^\bullet \longrightarrow I_C^\bullet$$

is **fully injective** if the three complexes above are injective resolutions of $\mathbf{Z}^q(C^\bullet)$, $\mathbf{B}^q(C^\bullet)$, and $H^q(C^\bullet)$, respectively. [2]

Lemma 2.4.1. Given a cochain complex $C^\bullet$, there exists a fully injective resolution of $C^\bullet$.

[2] In the literature, such a fully injective resolution is called a **Cartan–Eilenberg** resolution, but be aware that the conditions of this definition may vary from source to source.

Proof. Given a complex $C^\bullet$, we obtain the following short exact sequences:

$$0 \longrightarrow \mathbf{B}^q \longrightarrow \mathbf{Z}^q \longrightarrow H^q \longrightarrow 0,$$

$$0 \longrightarrow \mathbf{Z}^{q-1} \longrightarrow C^{q-1} \longrightarrow \mathbf{B}^q \longrightarrow 0.$$

We proceed inductively. We start by considering an injective resolution of

$$0 \longrightarrow \mathbf{Z}^{q-1} \longrightarrow C^{q-1} \longrightarrow \mathbf{B}^q \longrightarrow 0$$

using the Horseshoe Lemma:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}^{q-1} & \longrightarrow & C^{q-1} & \longrightarrow & \mathbf{B}^q & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & I^\bullet_{\mathbf{Z}^{q-1}} & \longrightarrow & I^\bullet_{C^{q-1}} & \longrightarrow & I^\bullet_{\mathbf{B}^q} & \longrightarrow & 0 \end{array}$$

Now consider an injective resolution of $0 \longrightarrow H^q \longrightarrow I^\bullet_{H^q}$. Also, considering now the short exact sequence :

$$0 \longrightarrow \mathbf{B}^q \longrightarrow \mathbf{Z}^q \longrightarrow H^q \longrightarrow 0,$$

and using again the Horseshoe Lemma for the injective resolutions of $\mathbf{B}^q$ and H^q defined before, we get :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{B}^q & \longrightarrow & \mathbf{Z}^q & \longrightarrow & H^q & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & I^\bullet_{\mathbf{B}^q} & \longrightarrow & I^\bullet_{\mathbf{Z}^q} & \longrightarrow & I^\bullet_{H^q} & \longrightarrow & 0 \end{array}$$

Gluing the previous digrams we get a unified diagram as follows :

$$\begin{array}{ccccccccccccc} C^q & \longrightarrow & \mathbf{B}^{q+1} & \longrightarrow & \mathbf{Z}^{q+1} & \longrightarrow & C^{q+1} & \longrightarrow & \mathbf{B}^{q+2} & \longrightarrow & \mathbf{Z}^{q+2} & \longrightarrow & C^{q+2} \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ I^\bullet_{C^q} & \longrightarrow & I^\bullet_{\mathbf{B}^{q+1}} & \longrightarrow & I^\bullet_{\mathbf{Z}^{q+1}} & \longrightarrow & I^\bullet_{C^{q+1}} & \longrightarrow & I^\bullet_{\mathbf{B}^{q+2}} & \longrightarrow & I^\bullet_{\mathbf{Z}^{q+2}} & \longrightarrow & I^\bullet_{C^{q+2}} \end{array}$$

gives the essential chain condition

$$I^{\bullet,q} \longrightarrow I^{\bullet,q+1} \longrightarrow I^{\bullet,q+2} = 0,$$

where $I^{\bullet,q} := I^\bullet_{C^q}$. So $I^{\bullet,\bullet}$ is a double complex of injective objects such that $0 \longrightarrow C^q \longrightarrow I^{\bullet,q}$ is an injective resolution. We show that $I^{\bullet,\bullet} := I^{\bullet,\bullet}_C$ is the desired fully injective resolution. We now have to show that we obtain injective resolutions as follows:

$$0 \longrightarrow \mathbf{Z}^q(C^\bullet) \longrightarrow \mathbf{Z}(\mathrm{d}_v^{0,q}) \longrightarrow \mathbf{Z}(\mathrm{d}_v^{1,q}) \longrightarrow \cdots$$

$$0 \longrightarrow \mathbf{B}^q(C^\bullet) \longrightarrow \mathbf{B}\left(\mathrm{d}_v^{0,q-1}\right) \longrightarrow \mathbf{B}\left(\mathrm{d}_v^{0,q-1}\right) \longrightarrow \cdots$$

$$0 \longrightarrow H^d\left(C^\bullet\right) \longrightarrow H^q\left(I^{0,\bullet}\right) \longrightarrow H^q\left(I^{1,\bullet}\right) \longrightarrow \cdots$$

By showing

$$I^{\bullet}_{\mathbf{Z}^d} = \mathbf{Z}\left(\mathrm{d}_v^{\bullet,q}\right), \quad I^{p}_{\mathbf{B}^d} = \mathbf{B}\left(\mathrm{d}_v^{\bullet,q-1}\right), \quad I^{\bullet}_{H^d} = H\left(I^{\bullet,q}\right),$$

we are done. We show the kernel case just for illustration. For all $p > 0$, we want to show $I^{p}_{\mathbf{Z}^d} = \mathbf{Z}\left(\mathrm{d}_v^{p,q}\right)$, where

$$\mathbf{Z}\left(\mathrm{d}_v^{p,q}\right) = \ker\left(I^{p,q} \xrightarrow{\mathrm{d}_{ver}^{p,q}} I^{p,q+1}\right).$$

By construction we have that :

$$\mathrm{d}_v^{p,q} = \mathrm{d}^{p,q} = I^{p}_{C^q} \longrightarrow I^{p}_{\mathbf{B}^{q+1}} \hookrightarrow I^{p}_{\mathbf{Z}^{q+1}} \hookrightarrow I^{p}_{C^{q+1}}.$$

Since the morphism $I^{p}_{\mathbf{B}^{d+1}} \hookrightarrow I^{p}_{\mathbf{Z}^{d+1}} \hookrightarrow I^{p}_{C^{q+1}}$ is a monomorphism, we get

$$\mathbf{Z}\left(\mathrm{d}_v^{p,q}\right) = \ker\left(I^{p}_{C^q} \longrightarrow I^{p}_{\mathbf{B}^{d+1}}\right) = I^{p}_{\mathbf{Z}^d}.$$

Finally, looking at the diagram 2.3, using the uniqueness of the induced map, we obtain that the induced complex on kernels is precisely $I^{\bullet}_{\mathbf{Z}^d}$ and we are done. Follow the same strategy to show

$$I^{\bullet}_{\mathbf{B}^d} = \mathbf{B}\left(\mathrm{d}_v^{\bullet,q-1}\right) \quad \text{and} \quad I^{\bullet}_{H^d} = H^q\left(I^{\bullet,q}\right).$$

□

Remark 2.4.2. An obvious idea for starting this proof is the following. Consider the diagram

$$\begin{array}{ccc} C^q & \longrightarrow & I^{\bullet}_{C^q} \\ \downarrow & & \\ C^{q+1} & \longrightarrow & I^{\bullet}_{C^{q+1}} \end{array}$$

for some chosen injective resolutions of C^q and C^{q+1} respectively. Then there is a lifting

$$\begin{array}{ccc} C^q & \longrightarrow & I^{\bullet}_{C^q} \\ \downarrow & & \vdots \\ C^{q+1} & \longrightarrow & I^{\bullet}_{C^{q+1}} \end{array}$$

However, the composition

$$I^{\bullet}_{C^{q-1}} \longrightarrow I^{\bullet}_{C^q} \longrightarrow I^{\bullet}_{C^{q+1}}$$

is not necessarily 0, even if $C^{d-1} \longrightarrow C^d \longrightarrow C^{d+1} = 0$. We only know that this complex morphism is null-homotopic. Nevertheless, even if it was 0, we could not ensure the existence of injective resolutions on the kernels or the images, necessarily, because a subobject of an injective object is not injective in general. Even more, the existence of such morphism of complexes does not even induce **a resolution** in general. Hence, it is natural to consider the following short exact sequences:

$$0 \longrightarrow \mathbf{Z}^q \longrightarrow C^q \longrightarrow \mathbf{B}^{q+1} \longrightarrow 0,$$

and

$$0 \longrightarrow \mathbf{B}^{q+1} \longrightarrow \mathbf{Z}^{q+1} \longrightarrow H^{q+1} \longrightarrow 0,$$

in order to preserve exactness at the level of injective resolutions using the Horseshoe Lemma.

2.4.2 The Grothendieck Spectral Sequence

Theorem 2.4.1 (Grothendieck Spectral Sequence/ Cohomological Version)**.** Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be abelian categories such that $\mathscr{A}$ and $\mathscr{B}$ have enough injectives. Let

$$G : \mathscr{A} \longrightarrow \mathscr{B} \quad \text{and} \quad F : \mathscr{B} \longrightarrow \mathscr{C}$$

be left exact functors. Suppose that G sends injective objects of $\mathscr{A}$ to F-acyclic objects of $\mathscr{B}$. Then, for every object A of $\mathscr{A}$, there exists a first quadrant cohomological spectral sequence

$$E_2^{p,q} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

Proof. Let A be an object of $\mathscr{A}$. Since $\mathscr{A}$ has enough injectives, we choose an injective resolution

$$0 \longrightarrow A \longrightarrow I^\bullet.$$

Applying G, we obtain a cochain complex in $\mathscr{B}$:

$$G(I^\bullet) : G(I^0) \xrightarrow{G(\mathrm{d}^0)} G(I^1) \xrightarrow{G(\mathrm{d}^1)} G(I^2) \longrightarrow \cdots .$$

Since G is left exact and $I^\bullet$ is an injective resolution of A, we have $H^q(G(I^\bullet)) \cong R^qG(A)$. Now, by Lemma 2.4.1, we choose a fully injective resolution of the complex $G(I^\bullet)$. Thus we have a double complex $J^{\bullet,\bullet}$ together with an augmentation $0 \longrightarrow G(I^\bullet) \longrightarrow J^{\bullet,\bullet}$. For every q, the horizontal complex

$$0 \longrightarrow G(I^q) \longrightarrow J^{0,q} \xrightarrow{\mathrm{d}_h^{0,q}} J^{1,q} \xrightarrow{\mathrm{d}_h^{1,q}} J^{2,q} \longrightarrow \cdots$$

is an injective resolution of $G\left(I^q\right)$. Moreover, since $J^{\bullet,\bullet}$ is fully injective, for every q the horizontal complex obtained by taking vertical cohomology,

$$0 \longrightarrow H^q\left(G(I^\bullet)\right) \longrightarrow H^q\left(J^{0,\bullet}, \mathrm{d}_v\right) \longrightarrow H^q\left(J^{1,\bullet}, \mathrm{d}_v\right) \longrightarrow H^q(J^{2,\bullet}, \mathrm{d}_v) \longrightarrow \cdots ,$$

is an injective resolution of $H^q(G(I^\bullet))$. Hence, we get that

$$0 \longrightarrow R^qG(A) \longrightarrow H^q(J^{0,\bullet}, \mathrm{d}_v) \longrightarrow H^q(J^{1,\bullet}, \mathrm{d}_v) \longrightarrow H^q(J^{2,\bullet}, \mathrm{d}_v) \longrightarrow \cdots$$

is an injective resolution of $R^qG(A)$. Now we apply F to the double complex $J^{\bullet,\bullet}$. We obtain a double complex $F(J^{\bullet,\bullet})$ in $\mathscr{C}$, and we consider the total complex

$$\mathrm{Tot}(F(J^{\bullet,\bullet}))^n = \bigoplus_{p+q=n} F(J^{p,q}),$$

with total differential $\mathrm{d} = \mathrm{d}_h + (-1)^p\mathrm{d}_v$. We now use the two spectral sequences associated with the two filtrations of this total complex. First, consider the first filtration

$${}^1F^p\,\mathrm{Tot}(F(J^{\bullet,\bullet}))^n = \bigoplus_{i \geq p} F(J^{i,n-i}).$$

By the spectral sequence of a double complex, this gives ${}^1E_2^{p,q} = H_h^p H_v^q(F(J^{\bullet,\bullet}))$. By the fully injective property, the vertical cohomology complex

$$H^q(J^{\bullet,\bullet}, \mathrm{d}_v)$$

is an injective resolution of $H^q(G(I^\bullet))$. Therefore, applying F and taking horizontal cohomology gives

$${}^1E_2^{p,q} = (R^pF)\left(H^q(G(I^\bullet))\right).$$

Since $H^q(G(I^\bullet)) \cong R^qG(A)$, we obtain ${}^1E_2^{p,q} = (R^pF)(R^qG)(A)$. Now we use the second filtration

$${}^2F^q\,\mathrm{Tot}(F(J^{\bullet,\bullet}))^n = \bigoplus_{j\geq q} F(J^{n-j,j}).$$

By the spectral sequence of a double complex, we have ${}^2E_1^{p,q} = H^p(F(J^{\bullet,q}), \mathrm{d}_h)$. For fixed q, the horizontal complex

$$0 \longrightarrow G(I^q) \longrightarrow J^{0,q} \longrightarrow J^{1,q} \longrightarrow J^{2,q} \longrightarrow \cdots$$

is an injective resolution of $G(I^q)$. Hence

$${}^2E_1^{p,q} = (R^pF)(G(I^q)).$$

Since I^q is injective in $\mathscr{A}$, by assumption $G(I^q)$ is F-acyclic. Therefore

$$(R^pF)(G(I^q)) = 0 \quad \text{for all } p > 0 \quad \text{and} \quad \left(R^0F\right)(G(I^q)) = F(G(I^q)).$$

Thus

$${}^2E_1^{p,q} = \begin{cases} F(G(I^q)), & p = 0, \\ \\ 0, & p > 0. \end{cases}$$

The differential d_1 on this page is induced by the vertical differential, so on the non-zero column it is precisely the differential of the complex $F(G(I^\bullet))$. Therefore

$${}^2E_2^{0,q} = H^q(F(G(I^\bullet))), \qquad {}^2E_2^{p,q} = 0 \quad \text{for } p > 0.$$

Hence this spectral sequence collapses at the E_2-page by Lemma 1.4.2 follows that :

$$H^n\left(\mathrm{Tot}(F(J^{\bullet,\bullet}))\right) \cong H^n(F(G(I^\bullet))).$$

Since $I^\bullet$ is an injective resolution of A, the complex $F(G(I^\bullet))$ computes the right derived functors of FG. Therefore

$$H^n\left(\mathrm{Tot}(F(J^{\bullet,\bullet}))\right) \cong H^n(F(G(I^\bullet))) \cong R^n(FG)(A).$$

Finally, the first spectral sequence has

$${}^1E_2^{p,q} = (R^pF)(R^qG)(A),$$

and it converges to the cohomology of the same total complex. Using the identification above, we conclude that

$$E_2^{p,q} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

This is the desired Grothendieck spectral sequence. □

Theorem 2.4.2 (Grothendieck Spectral Sequence / Homological Version)**.** Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be abelian categories such that $\mathscr{A}$ and $\mathscr{B}$ have enough projectives. Let

$$G : \mathscr{A} \longrightarrow \mathscr{B} \quad \text{and} \quad F : \mathscr{B} \longrightarrow \mathscr{C}$$

be right exact functors. Suppose that G sends projective objects of $\mathscr{A}$ to F-acyclic objects of $\mathscr{B}$. Then, for every object A of $\mathscr{A}$, there exists a first quadrant homological spectral sequence

$$E^2_{p,q} = (L_pF)(L_qG)(A) \Rightarrow L_{p+q}(FG)(A).$$

Chapter 3

Cohomology of Lie Algebras

3.1 Lie Algebras

Definition 3.1.1. a. Let k be a field. A **nonassociative algebra** over k is a vector space A over k equipped with a bilinear map

$$A \times A \longrightarrow A, \quad (x, y) \mapsto xy,$$

called the **multiplication**, which is not assumed to be associative.

b. A **Lie algebra** over k is a nonassociative algebra $\mathfrak{g}$ whose bilinear multiplication, written

$$[\,,\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x, y) \mapsto [x, y]$$

and called the **Lie bracket**, satisfies the following properties for all $x, y, z \in \mathfrak{g}$:

i. **Antisymmetry:** $[x, x] = 0$, and hence $[x, y] = -[y, x]$.

ii. **Jacobi identity:** $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

c. An **ideal** of a Lie algebra $\mathfrak{g}$ is a k-submodule $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$.

d. Let $\mathfrak{g}$ and $\mathfrak{h}$ two Lie algebras. A k - linear map $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is called a **Lie algebras homomorphism** if

$$\varphi([x, y]) = [\varphi(x), \varphi(y)], \quad \text{for all} \quad x, y \in \mathfrak{g}.$$

Remark 3.1.1. Every ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is itself a Lie algebra with the induced bracket. Moreover, the quotient $\mathfrak{g}/\mathfrak{h}$ inherits a natural Lie algebra structure, defined by

$$[x + \mathfrak{h},\, y + \mathfrak{h}] := [x, y] + \mathfrak{h}.$$

This is well defined precisely because $\mathfrak{h}$ is an ideal.

Proposition 3.1.1. For every ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, there is a short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0.$$

Example 3.1.1. Let k be a field and A a k-algebra. Then the set of all $n \times n$ matrices with entries in A, denoted $\mathbb{M}_n(A)$, is naturally a k-algebra with the usual matrix addition and multiplication.

Define a new operation on $\mathbb{M}_n(A)$ by

$$[X, Y] := XY - YX, \qquad X, Y \in \mathbb{M}_n(A).$$

Then this operation is bilinear and satisfies:

1. $[X, X] = 0$,

2. the Jacobi identity.

Hence it defines a Lie algebra structure on $\mathbb{M}_n(A)$, denoted $\mathfrak{gl}_n(A)$, called the **general linear Lie algebra**. Consider the subset

$$\mathfrak{sl}_n(A) := \{X \in \mathfrak{gl}_n(A) \mid \mathrm{tr}(X) = 0\}.$$

Since for all $X, Y \in \mathbb{M}_n(A)$ one has $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, it follows that $\mathrm{tr}([X, Y]) = 0$. Therefore $\mathfrak{sl}_m(A)$ is closed under the Lie bracket and is in fact an ideal of $\mathfrak{gl}_n(A)$. We thus obtain a short exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{sl}_n(A) \longrightarrow \mathfrak{gl}_n(A) \longrightarrow \frac{\mathfrak{gl}_n(A)}{\mathfrak{sl}_n(A)} \longrightarrow 0.$$

Example 3.1.2. Using the notation of Example 3.1.1, consider the subset of $\mathbb{M}_n(A)$ consisting of strictly upper triangular matrices. Denote this set by $\mathfrak{t}_n(A)$. One checks that:

- $\mathfrak{t}_m(A)$ is closed under the Lie bracket,

- for any $X \in \mathfrak{gl}_n(A)$ and $Y \in \mathfrak{t}_m(A)$, the commutator $[X, Y]$ is again strictly upper triangular.

Thus $\mathfrak{t}_m(A)$ is an ideal of $\mathfrak{gl}_m(A)$. Consequently, we obtain another short exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{t}_n(A) \longrightarrow \mathfrak{gl}_n(A) \longrightarrow \frac{\mathfrak{gl}_n(A)}{\mathfrak{t}_n(A)} \longrightarrow 0.$$

3.2 $\mathfrak{g}$-modules

Let $\mathfrak{g}$ be a Lie algebra over a field K.

Definition 3.2.1 ($\mathfrak{g}$-module)**.** A (left) **$\mathfrak{g}$-module** M is a k-vector space equipped with a k-bilinear map

$$\mathfrak{g} \times M \longrightarrow M, \quad (x, m) \mapsto x \cdot m,$$

such that for all $x, y \in \mathfrak{g}$ and $m \in M$,

$$[x, y] \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m).$$

Definition 3.2.2 (Homomorphism of $\mathfrak{g}$-modules)**.** Let M and N be $\mathfrak{g}$-modules. A **homomorphism of $\mathfrak{g}$-modules** $f : M \longrightarrow N$ is a k - linear map such that

$$f(x \cdot m) = x \cdot f(m), \quad \text{for all} \quad x \in \mathfrak{g},\ m \in M.$$

We denote by $\mathrm{Hom}_{\mathfrak{g}}(M, N)$ the set of all $\mathfrak{g}$-module homomorphisms from M to N.

Remark 3.2.1. If $a \in k$ and $f \in \mathrm{Hom}_{\mathfrak{g}}(M, N)$, then af is also a $\mathfrak{g}$-module homomorphism. Hence $\mathrm{Hom}_{\mathfrak{g}}(M, N)$ is a k-vector subspace of $\mathrm{Hom}_k(M, N)$. The category of (left) $\mathfrak{g}$-modules with these morphisms is denoted by $\mathfrak{g}$-mod.

Definition 3.2.3.

a. Let M be a $\mathfrak{g}$-module. The **invariant subspace** of M is

$$M^{\mathfrak{g}} := \{m \in M \mid x \cdot m = 0 \text{ for all } x \in \mathfrak{g}\}.$$

Viewing K as a $\mathfrak{g}$-module with the trivial action

$$x \cdot a = 0 \quad \text{for all } x \in \mathfrak{g},\ a \in K,$$

we have a natural isomorphism $M^{\mathfrak{g}} \cong \mathrm{Hom}_{\mathfrak{g}}(K, M)$.

b. Let M be a $\mathfrak{g}$-module. The space of **coinvariants** is defined as $M_{\mathfrak{g}} := M/\mathfrak{g}M$, where

$$\mathfrak{g}M := \mathrm{span}_k\{x \cdot m \mid x \in \mathfrak{g},\ m \in M\}.$$

Proposition 3.2.1. Let $\mathfrak{g}$ be a Lie algebra over a field K. Consider the functor

$$\mathrm{Tr} : k\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}$$

which assigns to a vector space V the trivial $\mathfrak{g}$-module structure defined by $x \cdot v = 0$ for all $x \in \mathfrak{g}$.

a. The invariants functor
$$(-)^{\mathfrak{g}} : \mathfrak{g}\text{-mod} \longrightarrow K\text{-mod}$$
is right adjoint to Triv. That is, for every $\mathfrak{g}$-module M and every k-vector space V, there is a natural isomorphism
$$\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Tr}(V), M) \cong \mathrm{Hom}_k(V, M^{\mathfrak{g}}).$$
In particular, $(-)^{\mathfrak{g}}$ is left exact.

b. The coinvariants functor
$$(-)_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \longrightarrow K\text{-mod}$$
is left adjoint to Tr. That is, for every $\mathfrak{g}$-module M and every k-vector space V, there is a natural isomorphism
$$\mathrm{Hom}_K(M_{\mathfrak{g}}, V) \cong \mathrm{Hom}_{\mathfrak{g}}(M, \mathrm{Tr}(V)).$$
In particular, $(-)_{\mathfrak{g}}$ is right exact.

Sketch of proof. a. Let $f : \mathrm{Tr}(V) \to M$ be a $\mathfrak{g}$-module homomorphism, where V carries the trivial action. Then, for all $v \in V$ and $x \in \mathfrak{g}$ we get :
$$x \cdot f(v) = f(x \cdot v) = f(0) = 0$$
Therefore $\mathrm{Im}\, f \subseteq M_{\mathfrak{g}}$, thus f induces a map $f \colon V \to M_{\mathfrak{g}}$. This observation defines an abelian group isomorphism as follows :
$$\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{Tr}(V), M\right) \cong \mathrm{Hom}_k\left(V, M_{\mathfrak{g}}\right)$$
for all $V \in \mathrm{Obj}(k - \text{mod})$ and $M \in \mathrm{Obj}\left(\mathfrak{g} - \text{mod}\right)$

b. Similarly to part (a).

□

Remark 3.2.2. We will see in the next section that the category $\mathfrak{g}$-mod has enough projective and injective objects. Therefore, the left and right derived functors of the coinvariants and invariants functors are well defined.

Definition 3.2.4. Let M be a $\mathfrak{g}$-module.

a. We write
$$H_*(\mathfrak{g}, M) \quad \text{or} \quad H_*^{\mathrm{Lie}}(\mathfrak{g}, M)$$
for the left derived functors $L_*\big(M \mapsto M_{\mathfrak{g}}\big)$ of the coinvariants functor, and call them the **homology groups** of $\mathfrak{g}$ with coefficients in M. By definition,
$$H_0(\mathfrak{g}, M) = M_{\mathfrak{g}}.$$

b. Similarly, we write

$$H^*(\mathfrak{g}, M) \quad \text{or} \quad H^*_{\mathrm{Lie}}(\mathfrak{g}, M)$$

for the right derived functors $R^*\big(M \mapsto M^{\mathfrak{g}}\big)$ of the invariants functor, and call them the **cohomology groups** of $\mathfrak{g}$ with coefficients in M. By definition,

$$H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}.$$

Example 3.2.1. a. Let $\mathfrak{g} = 0$. Then for every $\mathfrak{g}$-module M we have $M^{\mathfrak{g}} = M$ and $M_{\mathfrak{g}} = M$. Thus both the invariants and coinvariants functors are exact. Consequently, all higher derived functors vanish, and we obtain

$$H_n(0, M) = 0 \quad \text{and} \quad H^n(0, M) = 0 \quad \text{for all } n > 0.$$

b. Let $\mathfrak{g}$ be the free k-module with basis $e_1, \dots, e_n$, endowed with the structure of an abelian Lie algebra (i.e. $[e_i, e_j] = 0$ for all i, j). Then a $\mathfrak{g}$-module is simply a k-module M equipped with n commuting endomorphisms $e_1, \dots, e_n$ acting on M. Equivalently, the category $\mathfrak{g}$-mod is isomorphic to the category of modules over the polynomial ring

$$R = k[e_1, \dots, e_n].$$

Let k be the trivial $\mathfrak{g}$-module. Viewed as an R-module, this corresponds to the structure where each e_i acts as zero. Then, for any $\mathfrak{g}$-module M, we have

$$M_{\mathfrak{g}} \cong K \otimes_R M, \qquad M^{\mathfrak{g}} \cong \mathrm{Hom}_R(k, M).$$

Therefore, the Lie algebra homology and cohomology of $\mathfrak{g}$ with coefficients in M can be identified with the usual derived functors:

$$H^{\mathrm{Lie}}_*(\mathfrak{g}, M) \cong \mathrm{Tor}^R_*(k, M), \qquad H^*_{\mathrm{Lie}}(\mathfrak{g}, M) \cong \mathrm{Ext}^*_R(k, M).$$

3.3 The Hochschild–Serre spectral sequence

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ an ideal of $\mathfrak{g}$. Then the quotient $\mathfrak{g}/\mathfrak{h}$ inherits a natural Lie algebra structure, and we have a short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0.$$

Lemma 3.3.1. Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$, and let M be a $\mathfrak{g}$-module. Then $M^{\mathfrak{h}}$ and $M_{\mathfrak{h}}$ are naturally $\mathfrak{g}/\mathfrak{h}$-modules. Moreover, if

$$\rho : (\mathfrak{g}/\mathfrak{h})\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}$$

is the forgetful functor induced by the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, then ρ has $(-)_{\mathfrak{h}}$ as a left adjoint and $(-)^{\mathfrak{h}}$ as a right adjoint.

Proof. We first define the action of $\mathfrak{g}/\mathfrak{h}$ on $M^{\mathfrak{h}}$. For $x \in \mathfrak{g}$ and $m \in M^{\mathfrak{h}}$, set $\overline{x}m := xm$, where $\overline{x}$ denotes the class of x in $\mathfrak{g}/\mathfrak{h}$. This action is clearly well defined, and the induced map

$$\mathfrak{g}/\mathfrak{h} \times M^{\mathfrak{h}} \to M^{\mathfrak{h}}, \quad (\overline{x}, m) \mapsto \overline{x} \cdot m$$

can be easily shown to be bilinear and to satisfy the property

$$[\overline{x}, \overline{y}] \cdot m = \overline{x} \cdot (\overline{y} \cdot m) - \overline{y} \cdot (\overline{x} \cdot m).$$

Similarly, $M_{\mathfrak{h}}$ becomes a $\mathfrak{g}/\mathfrak{h}$-module by $\overline{x} \cdot \overline{m} := \overline{xm}$, where $\overline{m}$ denotes the class of m in $M_{\mathfrak{h}}$. Now let $M \in \mathfrak{g}$-mod and $N \in (\mathfrak{g}/\mathfrak{h})$-mod. We prove the two adjunctions. First, we construct a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}(M_{\mathfrak{h}}, N) \cong \mathrm{Hom}_{\mathfrak{g}}(M, \rho N).$$

Let $\varphi \in \mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}(M_{\mathfrak{h}}, N)$. We define the following morphism:

$$\Phi(\varphi)\colon M \to \rho(N), \quad \Phi(\varphi)(m) = \varphi(\overline{m}).$$

Conversely, for $\psi \in \mathrm{Hom}_{\mathfrak{g}}(M, \rho N)$ we consider the morphism:

$$\Psi(\psi)\colon M_{\mathfrak{h}} \to N, \quad \Psi(\psi)(\overline{m}) = \psi(m).$$

Show that these two maps

$$\Phi\colon \mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}(M_{\mathfrak{h}}, N) \to \mathrm{Hom}_{\mathfrak{g}}(M, \rho N) \quad \text{and} \quad \Psi\colon \mathrm{Hom}_{\mathfrak{g}}(M, \rho N) \to \mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}(M_{\mathfrak{h}}, N)$$

are well defined and are inverse to each other, and therefore $\mathrm{Hom}_{\mathfrak{g}}(M, \rho N) \cong \mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}(M_{\mathfrak{h}}, N)$. It is left as an exercise to show that these isomorphisms are natural, i.e. that for $M_1, M_2, M \in \mathfrak{g}$-mod and $N_1, N_2, N \in (\mathfrak{g}/\mathfrak{h})$-mod and any choice

$$f\colon M_1 \to M_2 \in \mathrm{Hom}_{\mathfrak{g}}(M_1, M_2) \quad \text{and} \quad g\colon N_1 \to N_2 \in \mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}(N_1, N_2),$$

the following squares commute:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}\left((M_2)_{\mathfrak{h}}, N\right) & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathfrak{g}}(M_2, \rho N) \\ \big\downarrow{\scriptstyle -\circ f_{\mathfrak{h}}} & & \big\downarrow{\scriptstyle -\circ f} \\ \mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}\left((M_1)_{\mathfrak{h}}, N\right) & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathfrak{g}}(M_1, \rho N) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}\big(M_{\mathfrak{h}}, N_1\big) & \xrightarrow{\quad\Phi\quad} & \mathrm{Hom}_{\mathfrak{g}}\big(M, \rho N_1\big) \\ {\scriptstyle g\circ -}\big\downarrow & & \big\downarrow{\scriptstyle \rho(g)\circ -} \\ \mathrm{Hom}_{\mathfrak{g}/\mathfrak{h}}\big(M_{\mathfrak{h}}, N_2\big) & \xrightarrow{\quad\Phi\quad} & \mathrm{Hom}_{\mathfrak{g}}\big(M, \rho N_2\big). \end{array}$$

Thus $(-)_{\mathfrak{h}}$ is left adjoint to ρ. $\square$

Exercise 3.3.1. Show that the functor $(-)^\natural$ is right adjoint to ρ.

Lemma 3.3.2. Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. Let

$$F : \mathscr{A} \longrightarrow \mathscr{B} \qquad \text{and} \qquad G : \mathscr{B} \longrightarrow \mathscr{A}$$

be additive functors such that F is right adjoint to G. Consider the following conditions:

a. G transforms monomorphisms into monomorphisms,

b. G is exact,

c. F transforms injective objects into injective objects.

Then $(a) \Leftrightarrow (b) \Rightarrow (c)$. If $\mathscr{A}$ has enough injectives, then all three conditions are equivalent.

Proof. Since G is a left adjoint, it is right exact. Hence G is exact if and only if it preserves monomorphisms. Thus $(a) \Leftrightarrow (b)$. Assume now that (a) holds. Let I be an injective object of $\mathscr{A}$. We show that $F(I)$ is injective in $\mathscr{B}$. Consider the following diagram

$$\begin{array}{ccc} N & \overset{\varphi}{\hookrightarrow} & M \\ \downarrow{\scriptstyle \alpha} & & \\ F(I) & & \end{array}$$

By adjunction, α corresponds to a morphism $\tilde{\alpha} : G(N) \to I$. By assumption, $G(\varphi) : G(N) \to G(M)$ is a monomorphism. Since I is injective, there exists a morphism $\tilde{\beta} : G(M) \longrightarrow I$ extending $\tilde{\alpha}$. By adjunction again, $\tilde{\beta}$ corresponds to a morphism $\beta : M \longrightarrow F(I)$ extending α. Therefore $F(I)$ is injective. Hence $(b) \Rightarrow (c)$. Finally, assume that $\mathscr{A}$ has enough injectives and that (c) holds. We show that G preserves monomorphisms. Let $f : B \longrightarrow B'$ be a monomorphism in $\mathscr{B}$, and set $A = \ker(G(f))$. Since $\mathscr{A}$ has enough injectives, we can choose a monomorphism

$$g : A \hookrightarrow I$$

with I injective. By construction, $A \to G(B)$ is the kernel of $G(f)$. Since I is injective, the morphism $g : A \to I$ extends to a morphism $\tilde{g} : G(B) \longrightarrow I$. By adjunction, $\tilde{g}$ corresponds to a morphism $B \to F(I)$. By assumption, $F(I)$ is injective. Since $f : B \to B'$ is a monomorphism, this morphism extends to a morphism $h : B' \longrightarrow F(I)$ making the following square commutes :

$$\begin{array}{ccc} B & \overset{f}{\longrightarrow} & B' \\ \downarrow & \swarrow{\scriptstyle h} & \\ F(I) & & \end{array}$$

By adjunction, h corresponds to a morphism $\tilde{h} : G(B') \longrightarrow I$. Then the composition

$$G(B) \longrightarrow G(B') \xrightarrow{h'} I$$

is equal to $\tilde{g}$. In particular, restricted to A, it agrees with g. But A maps to zero in $G(B')$, because $A = \ker(v(f))$. Hence the restriction of this composition to A is zero. Therefore $g = 0$. Since g is a monomorphism, this forces $A = 0$. Hence $\ker(G(f)) = 0$, so $G(f)$ is a monomorphism. Thus (a) holds. $\square$

Corollary 3.3.1. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h} \subseteq \mathfrak{g}$ be an ideal. Then the functor

$$(-)^{\mathfrak{h}} : \mathfrak{g}\text{-mod} \longrightarrow (\mathfrak{g}/\mathfrak{h})\text{-mod}$$

sends injective objects to injective objects. Moreover, the functor

$$(-)^{\mathfrak{g}} : \mathfrak{g}\text{-mod} \longrightarrow k\text{-mod}$$

also sends injective objects to injective objects.

Proof. The desired result comes immediately using Lemma 3.3.2, Proposition 3.2.1 and Lemma 3.3.1. $\square$

Theorem 3.3.1 (Hochschild–Serre Spectral Sequences)**.** Let $\mathfrak{h}$ be an ideal of a Lie algebra $\mathfrak{g}$. Then, for every $\mathfrak{g}$-module M, there exist two convergent first quadrant spectral sequences:

$$E^2_{p,q} = H_p\left(\mathfrak{g}/\mathfrak{h}, H_q(\mathfrak{h}, M)\right) \Rightarrow H_{p+q}(\mathfrak{g}, M),$$

and

$$E_2^{p,q} = H^p\left(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h}, M)\right) \Rightarrow H^{p+q}(\mathfrak{g}, M).$$

Proof. We prove the cohomological case. The homological case is analogous, using left derived functors instead of right derived functors. Let

$$\rho : (\mathfrak{g}/\mathfrak{h})\text{-mod} \longrightarrow \mathfrak{g}\text{-mod}$$

be the forgetful functor induced by the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. We claim that the invariants functors factor as follows:

$$\begin{array}{ccc} \mathfrak{g}\text{-mod} & \xrightarrow{(-)^{\mathfrak{h}}} & (\mathfrak{g}/\mathfrak{h})\text{-mod} \\ & {\scriptstyle (-)^{\mathfrak{g}}} \searrow & \downarrow {\scriptstyle (-)^{\mathfrak{g}/\mathfrak{h}}} \\ & & k\text{-mod} \end{array}$$

Using this observation, we shall use Theorem 2.4.1 considering the Grothendieck spectral sequence assosiated with the left exact functor $(-)^{\mathfrak{h}}$ and $(-)^{\mathfrak{g}/\mathfrak{h}}$. Doing this we obtain

$$E_2^{p,q} = H^p\left(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h}, M)\right) \Rightarrow R^{p+q}\left[\left((-)^{\mathfrak{g}/\mathfrak{h}} \circ (-)^{\mathfrak{h}}\right)\right](M) = R^{p+q}\left[(-)^{\mathfrak{g}}\right](M) = H^{p+q}(\mathfrak{g}, M).$$

The last thing that we have to show is that $(-)^{\mathfrak{h}}$ sends $\mathfrak{g}-\text{mod}$ objects to $(-)^{\mathfrak{g}/\mathfrak{h}}$ - acyclic elements. From Lemma 3.3.1 if $I \in \text{Obj}(\mathfrak{g}-\text{mod})$, then $I^{\mathfrak{h}}$ is injective in $\mathfrak{g}/\mathfrak{h}$, hence is $(-)^{\mathfrak{g}/\mathfrak{h}}$ - acyclic. □

Example 3.3.1. Let $\mathfrak{g} = \mathfrak{gl}_2(k)$ and $\mathfrak{h} = \mathfrak{sl}_2(k)$. Then from Example 3.1.1 we have that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and we have a short exact sequence

$$0 \longrightarrow \mathfrak{sl}_2(k) \longrightarrow \mathfrak{gl}_2(k) \longrightarrow \mathfrak{gl}_2(k)/\mathfrak{sl}_2(k) \longrightarrow 0.$$

Our goal is to find the cohomology groups $H^n\left(\mathfrak{gl}_2(k), k\right)$ using the Hochschild – Serre sequence. We first calculate the cohomology groups $H^n\left(\mathfrak{gl}_2(k)/\mathfrak{sl}_2(k), k\right)$ and $H^n\left(\mathfrak{sl}_2(k), k\right)$. Since $\dim_k \mathfrak{sl}_2(k) = 3$ and $\dim_k \mathfrak{gl}_2(k) = 4$, it follows that

$$\dim_k\big(\mathfrak{gl}_2(k)/\mathfrak{sl}_2(k)\big) = 1.$$

Hence, as a k-vector space, $\mathfrak{gl}_2(k)/\mathfrak{sl}_2(k) \cong k$, and since every 1-dimensional Lie algebra is abelian, the quotient is an abelian Lie algebra. Let k be the trivial $\mathfrak{g}$-module. By Example 3.2.1 (b),

$$H^n\big(\mathfrak{gl}_2(k)/\mathfrak{sl}_2(k), k\big) \cong \text{Ext}^n_{k[x]}(k, k).$$

Using the projective resolution

$$0 \longrightarrow k[x] \xrightarrow{\cdot x} k[x] \longrightarrow k \longrightarrow 0,$$

we obtain

$$H^0\big(\mathfrak{gl}_2(k)/\mathfrak{sl}_2(k), k\big) \cong k, \quad \text{and} \quad H^1\big(\mathfrak{gl}_2(k)/\mathfrak{sl}_2(k), k\big) \cong k,$$

and

$$H^n\big(\mathfrak{gl}_2(k)/\mathfrak{sl}_2(k), k\big) = 0 \quad \text{for all } n \geq 2.$$

On the other hand, by **Section 6.4** [2] we get that $\mathfrak{sl}_2(k)$ is simple, and using **Theorem 21.1** [3] and **Exercise 7.7.3** [4]

$$H^q(\mathfrak{sl}_2(k), k) = \begin{cases} k, & q = 0, \\ k, & q = 3, \\ 0, & \text{otherwise.} \end{cases}$$

We now consider the **Hochschild–Serre** spectral sequence :

$$E_2^{p,q} = H^p\big(\mathfrak{gl}_2(k)/\mathfrak{sl}_2(k),\, H^q(\mathfrak{sl}_2(k), k)\big) \;\Rightarrow\; H^{p+q}(\mathfrak{gl}_2(k), k).$$

Thus the only nonzero terms are

$$E_2^{0,0} = k, \quad E_2^{1,0} = k, \quad E_2^{0,3} = k, \quad E_2^{1,3} = k.$$

Hence we conclude

$$H^n(\mathfrak{gl}_2(k), k) \cong \begin{cases} k, & n = 0, \\ k, & n = 1, \\ 0, & n = 2, \\ k, & n = 3, \\ k, & n = 4, \\ 0, & n \geq 5. \end{cases}$$

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